1D and 2D symbolic dynamical systems

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Alphabets and Configurations

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- (dD) **Configuration**: A mapping $c\colon \mathbb{Z}^d \to \mathcal{A}$.
- (dD) **Full shift**: The set of all configurations $\mathcal{A}^{\mathbb{Z}^d}$.

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- **Shift action**: $\sigma^{\vec{n}}(c)(\vec{x}) = c(\vec{x}+\vec{n})$, $\forall c \in \mathcal{A}^{\mathbb{Z}^d}, \forall \vec{x} \in \mathbb{Z}^d$.

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- **Shift action**: $\sigma^{\vec{n}}(c)(\vec{x}) = c(\vec{x}+\vec{n})$, $\forall c\in\mathcal{A}^{\mathbb{Z}^d}, \forall \vec{x}\in\mathbb{Z}^d$.
- Shifts are homeomorphisms.

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- $X_{\mathcal{F}} = \{c \in \mathcal{A}^{\mathbb{Z}^d} : \sigma^{\vec{n}}(c)_{|D} \neq p\}$, for all $\vec{n} \in \mathbb{Z}^d$ and all patterns $p\colon D \to \mathcal{A}$ in $\mathcal{F}$.

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- Subshifts are the closed and σ-invariant subsets of $\mathcal{A}^{\mathbb{Z}^d}$.
- **Subshift of Finite Type (SFT)**: A subshift defined by a *finite* set of forbidden patterns.

1D SFTs

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- Perron-Frobenius Theory, principal eigenvalue, entropy.

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- Finite number of tiles, but infinite copies of each tile.
- Adjacent tiles must have the same color in abutting edges.
- No (efficient) algebraic description.

Aperiodic SFTs

- **Periodic configuration** c:
  $\exists$ lineraly independent $\vec{n_1}, \ldots, \vec{n_d} \in \mathbb{Z}^d$ such that $c(\vec{x} + \vec{n_i}) = c(\vec{x})$, $\forall \vec{x} \in \mathbb{Z}^d$ and $i = 1, \ldots, d$.

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- **Aperiodic SFT**: Non-empty, but does not contain a periodic configuration.
- No aperiodic 1D SFT. Cycle in the graph.

What happens in higher dimensions?

Theorem (Berger 1966, Robinson 1971, Kari-Culik 1995, Ollinger 2010, Jeandel-Rao 2015)

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- Smallest (possible) has 11.

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- If aperiodic SFTs do not exist, then (exactly) one of the semi-algorithms will halt.
- Emptiness is decidable for 1D SFTs.

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- Sierpinski carpet can be "realized".
- Embedding of Turing Machine computations inside.

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- $\mathcal{N}(X)$ denotes the set of *non*-expansive directions of X.

Extremely Expansive Subshifts

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- Are extremely expansive 2D SFTs closer to the 1D or to the 2D case?

Aperiodicity and Undecidabilty for Extremely Expansive SFTs

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- Extremely expansive remain essentially 2D.

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Theorem (Hochman 2011)

For every closed set of directions $\mathcal{N}_0$, there exists a subshift X such that $\mathcal{N}(X) = \mathcal{N}_0$.

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Theorem (Guillon-Z. 2016)

For every effectively closed set of directions $\mathcal{N}_0$, there exists a 2D SFT X such that $\mathcal{N}(X) = \mathcal{N}_0$.

Non-expansive direction of an extremely expansive 2D SFT

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- There exists an algorithm that takes n and gives an approximation of error $\leq 2^{-n}$.
- Connection between computational and geometric notions.

Thank you

Σας ευχαριστώ!