

An introduction to Discrete Dynamical systems

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*"Perhaps we would all be better off, not only in research and teaching, but also in every day political and economical life, if more people would take into consideration **that simple dynamical systems do not necessarily lead to simple dynamical behavior**"*

R.May, "Simple mathematical models with very complicated dynamics", Nature 261, 1976

A simple example

(Peitgen & Richter, 1984)

Deposit Z_0

Z_0, Z_1, Z_2, \dots

Rate of interest ε

$$Z_{n+1} = (1 + \varepsilon)Z_n = (1 + \varepsilon)^{n+1}Z_0$$

Period T

Prohibition of unlimited wealth \Rightarrow reduction of the rate of interest proportional to Z_n

$$\varepsilon = \varepsilon_0(1 - Z_n / Z_{\max})$$

$$Z_{n+1} = \varepsilon_0(1 - Z_n / Z_{\max})Z_n \quad \Rightarrow \quad Z_{n+1} = (1 + \varepsilon_0)(1 - x_n)Z_n$$

$$\frac{Z_n}{Z_{\max}} = \frac{1 + \varepsilon_0}{\varepsilon} x_n$$

¶

$$x_{n+1} = rx_n(1 - x_n)$$

Logistic map

$$r = 1 + \varepsilon_0$$

$$x_n = \frac{Z_n}{Z_{\max}} \frac{\varepsilon_0}{1 + \varepsilon_0}$$

[[orbits](#)]

A. 1D discrete maps

One Dimensional maps

$$x_{n+1} = f(x_n), \quad x_i \in R, \quad n = 0, 1, 2, \dots$$

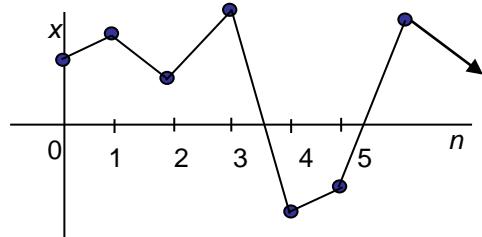
The sequence of values $T = \{x_0, x_1, x_2, \dots\}$ is a partial solution or a trajectory of the map that corresponds to the initial value x_0 .

$$x_1 = f(x_0)$$

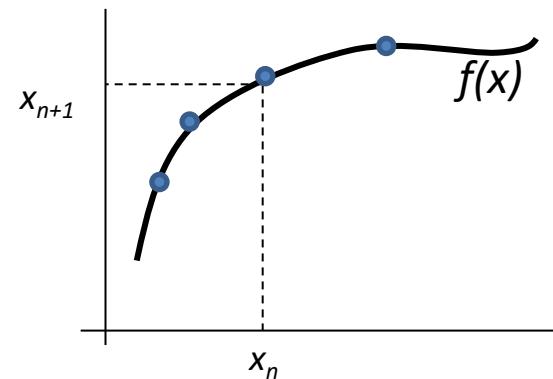
$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$

.....

$$x_n = f(x_{n-1}) = f(f(\dots f(x_0) \dots)) = f^n(x_0)$$



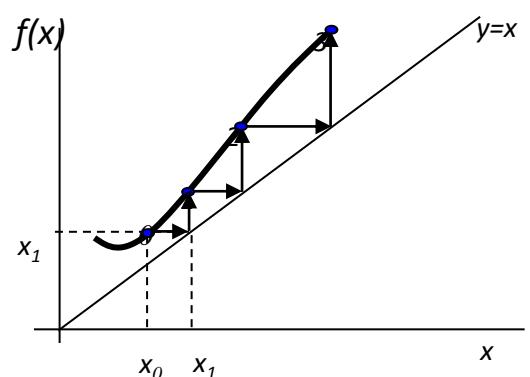
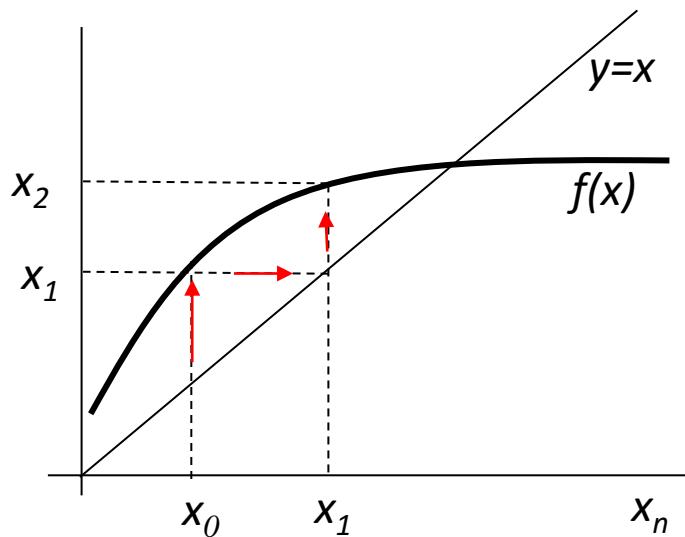
Plane of the map (x_n, x_{n+1})



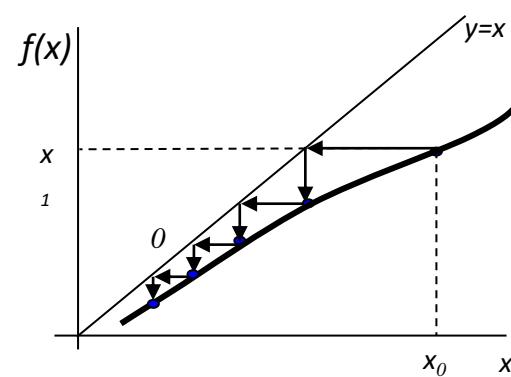
definitions

- The map is called **invertible** if there exist f^1 , i.e. $x_n = f^{-1}(x_{n+1})$
In this case $T=\{ \dots x_{-2}, x_{-1}, x_0, x_1, x_2, \dots \}$
- An interval $I \subset R$ is **invariant** under the map if $\forall x \in I, f(x) \in I$

Graphical presentation (cobweb)



$f(x) > x > 0$.
 $\lim x_n = \infty$ ($n \rightarrow \infty$)



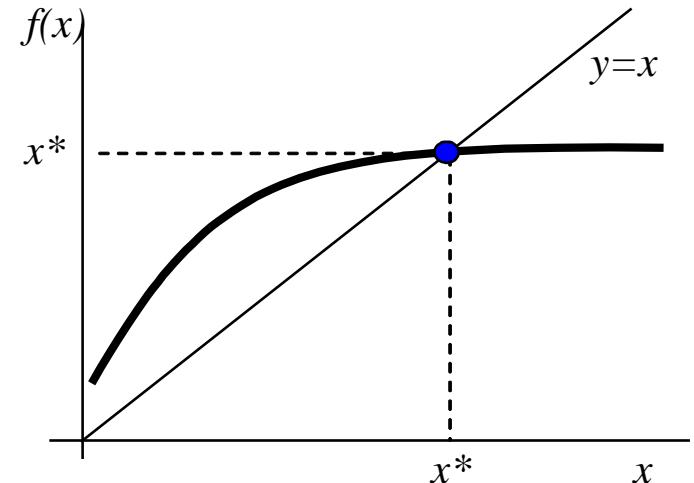
$0 < f(x) < x$.
 $\lim x_n = 0$ ($n \rightarrow \infty$)

Special solutions

- Equilibrium solution or **fixed point**

$$x_n = x^*, \quad \forall n$$

$$f(x^*) - x^* = 0$$

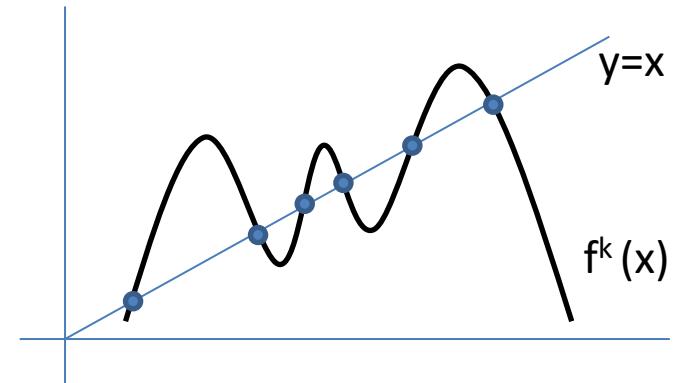


- Periodic orbits of period k

$$x_{n+k} = x_n, \quad \forall n \quad \Rightarrow \quad T^{(k)} = \{x_0, x_1, \dots, x_{k-1}, x_0, x_1, \dots\}$$

$$\text{if } x^* \in T^{(k)} \quad \Rightarrow \quad x^* = f^k(x^*)$$

$$f^k(x^*) - x^* = 0 \quad (m \cdot k \text{ solutions})$$



Linear maps

- Constant coefficients

$$x_{n+1} = ax_n + b \quad \Rightarrow \quad x_n = \begin{cases} x_0 + bn & \text{if } a = 1 \\ \left(x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a} & \text{if } a \neq 1 \end{cases}$$

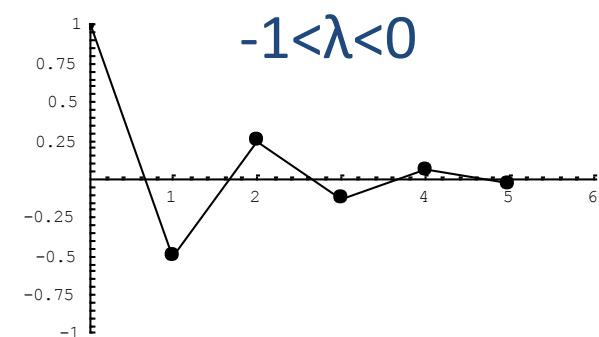
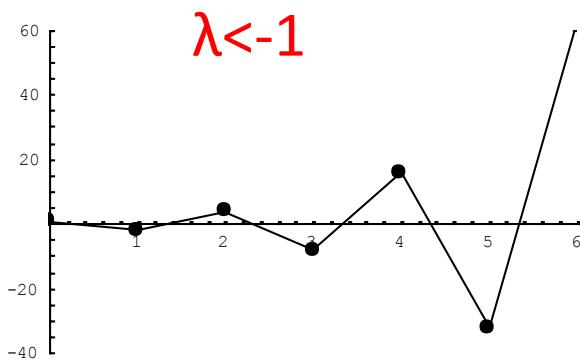
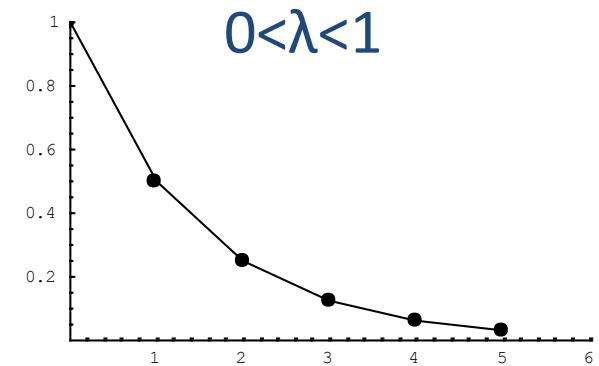
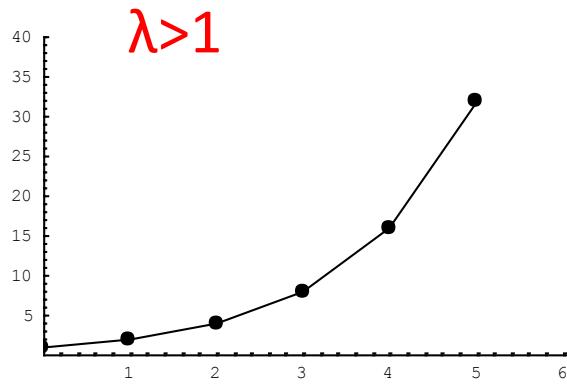
- Variable Coefficients

$$x_{n+1} = a_n x_n + b_n \quad \Rightarrow \quad x_n = \left(\prod_{i=0}^{n-1} a_i \right) x_0 + \sum_{k=0}^{n-1} \left(\prod_{i=k+1}^{n-1} a_i \right) b_k$$

- Mathematica command : **RSolve**

Linear maps

$$x_{n+1} = \lambda x_n \quad (\lambda \in R) \quad \Rightarrow \quad x_n = \lambda^n x_0$$



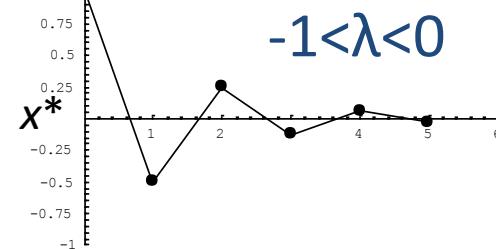
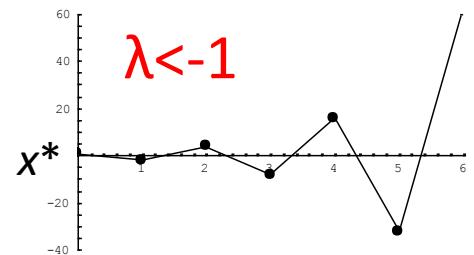
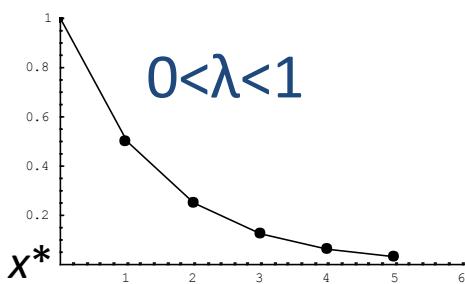
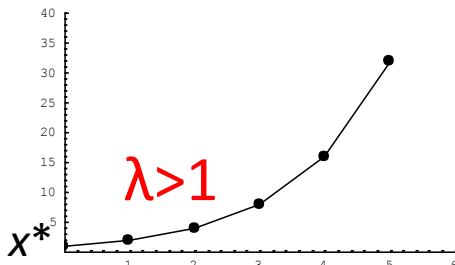
Linearization near Fixed Points

map $x_{n+1} = f(x_n)$

Fixed point $x^* = f(x^*)$

$$x_n = x^* + \delta_n, \quad x_{n+1} = x^* + \delta_{n+1}, \quad \delta_n \ll 1 \quad \forall n$$

$$x^* + \delta_{n+1} = f(x^* + \delta_n) = f(x^*) + \frac{df}{dx} \Big|_{x=x^*} \delta_n + O(\delta_n^2) \Rightarrow \delta_{n+1} = \lambda \delta_n, \quad \lambda = \frac{df}{dx} \Big|_{x=x^*} = f'(x^*)$$



Stability

$$\lambda = f'(x^*)$$

$\lambda > 1$: x^* is unstable

$0 < \lambda < 1$: x^* is asymptotically stable

$-1 < \lambda < 0$: x^* is asymptotically stable with reflection

$\lambda < -1$: x^* is unstable

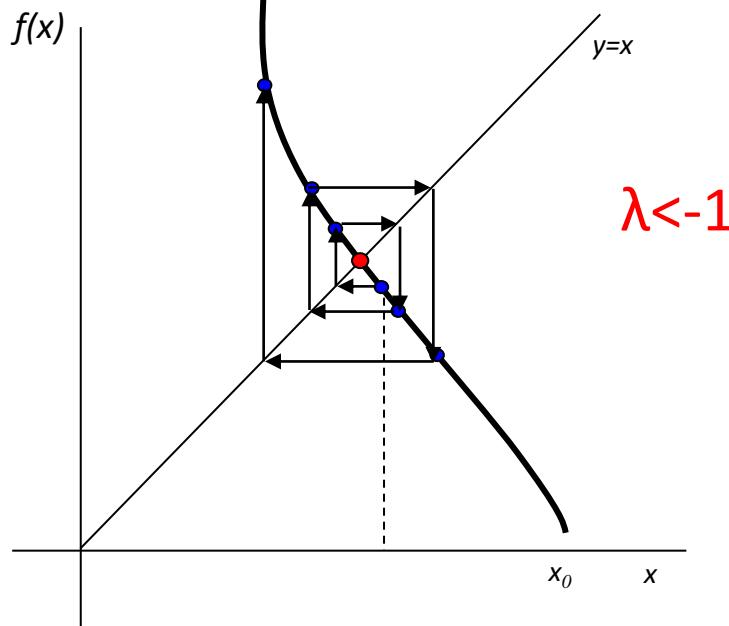
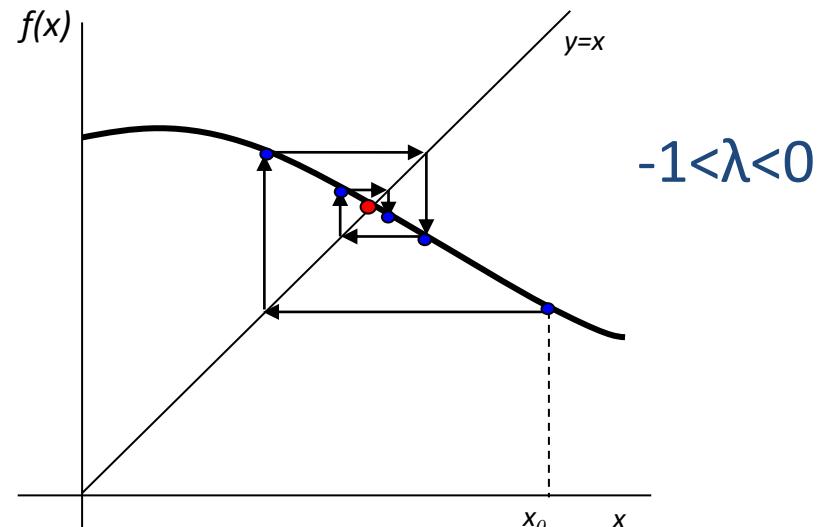
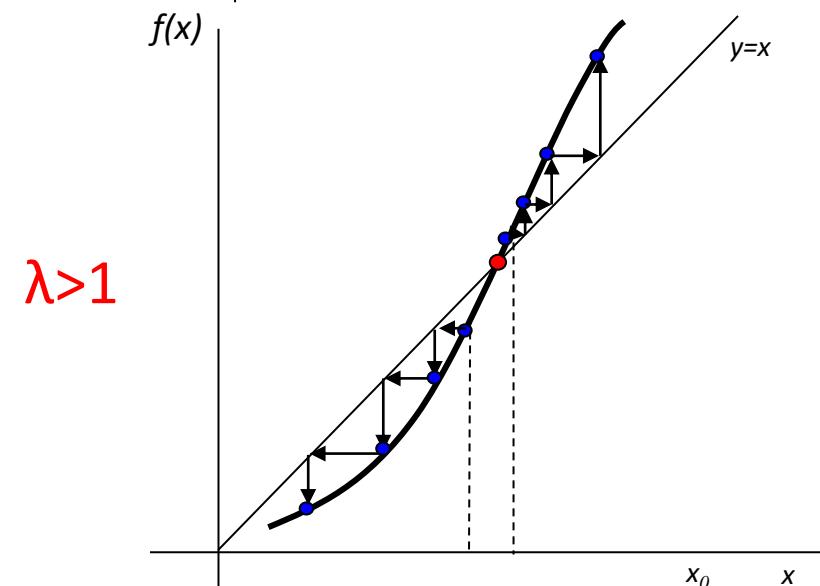
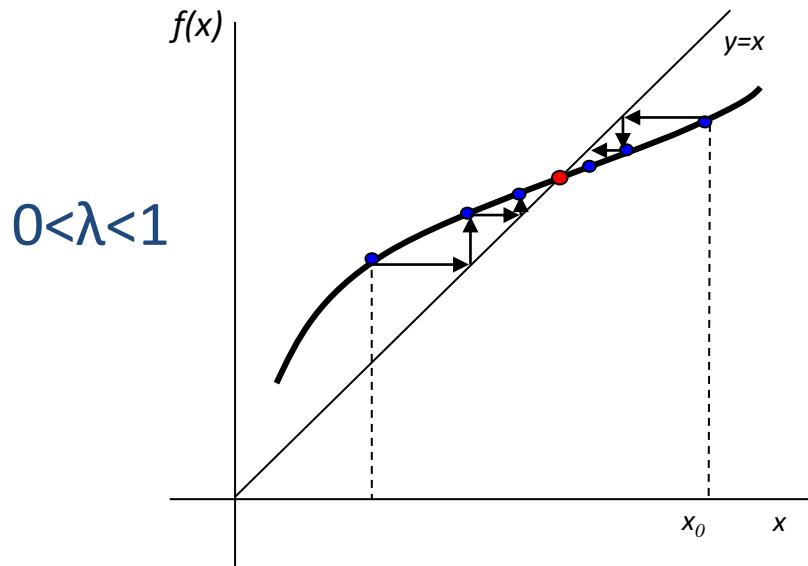
x^* = attractor or sink

x^* = repeller or source

If $|\lambda|=1$ then x^* is **parabolic** or **nonhyperbolic** (critical linear stability)

If $|\lambda| \neq 1$ then x^* is **hyperbolic**

Graphical presentation of stability



Stability in nonhyperbolic case

A. Case $f'(x^*)=1, f''(x^*)\neq 0$

- If $f''(x^*)<0$ then x^* is **semi-stable from above**
- If $f''(x^*)>0$ then x^* is **semi-stable from below**

B. Case $f'(x^*)=-1, f''(x^*)\neq 0$ [$s_f(x)=2f'''(x)+3(f''(x))^2$]

- If $s_f(x^*)>0$ then x^* is **asymptotically stable**
- If $s_f(x^*)<0$ then x^* is **unstable**

C. Case $f'(x^*)=1$ or $-1, f''(x^*)=0$

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

Schwarzian
derivative

- If $S_f(x^*)<0$ then x^* is **asymptotically stable**
- If $S_f(x^*)>0$ then x^* is **unstable**

Stability of periodic orbits

As for fixed points of the map

$$x_{n+1} = f^k(x_n)$$

$$\lambda = \left. \frac{df^k}{dx} \right|_{x=x_i^*}, \quad x_i^* \in T^{(k)} = \{x_0^*, x_1^*, \dots, x_{k-1}^*\}$$

$$\left. \frac{df^k}{dx} \right|_{x=x_0} = f'(x_{k-1})f'(x_{k-2}) \dots f'(x_1)f'(x_0) = \prod_{i=0}^{k-1} f'(x_i)$$

Bifurcations

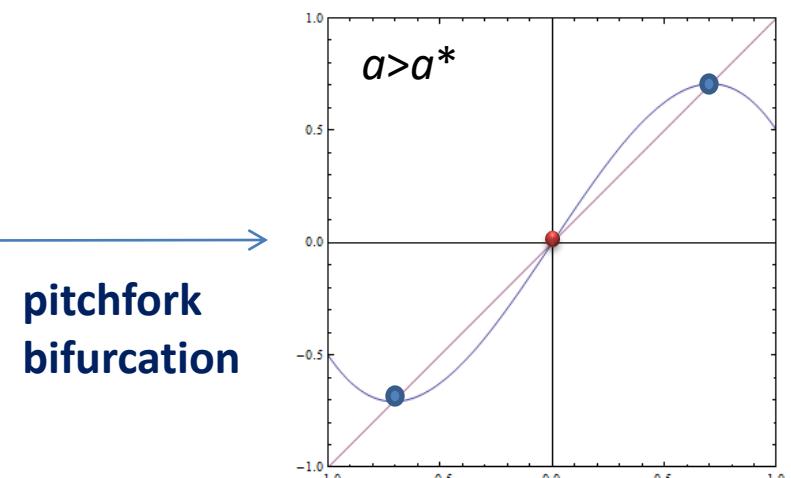
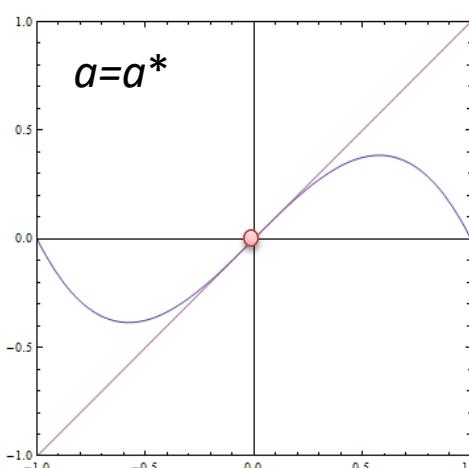
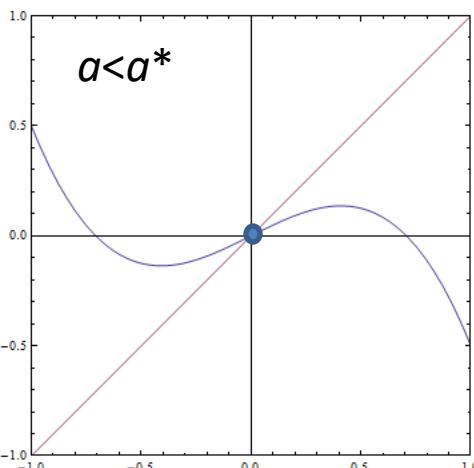
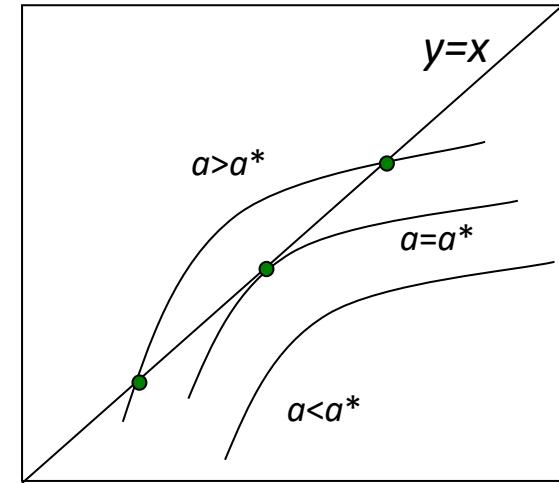
$$x_{n+1} = f(x_n; a), \quad a \in \mathbf{R}$$

parameter

Necessary condition

$$f'(x^*; a^*) = 1$$

Tangent bifurcation



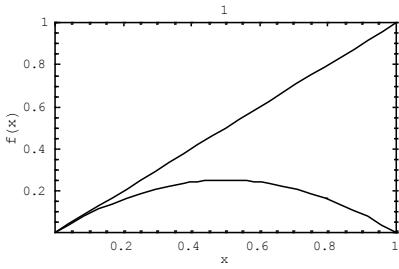
pitchfork
bifurcation

* If $f'(x^*; a^*) = -1$ then $f''(x^*; a^*) = 1 \Rightarrow$ pitchfork bifurcation with period doubling

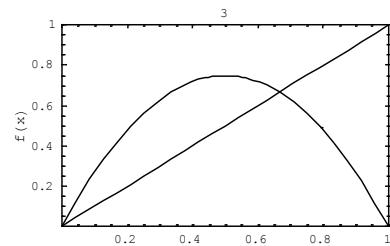
The logistic map

$$f(x) = r x(1-x) \quad , \quad x_{n+1} = r x_n (1-x_n), \quad r > 0, \quad x_n \in R$$

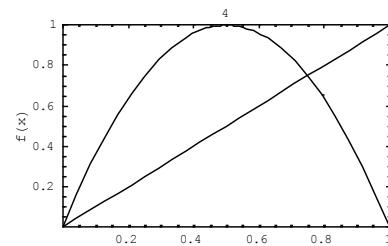
$0 < r < 1$



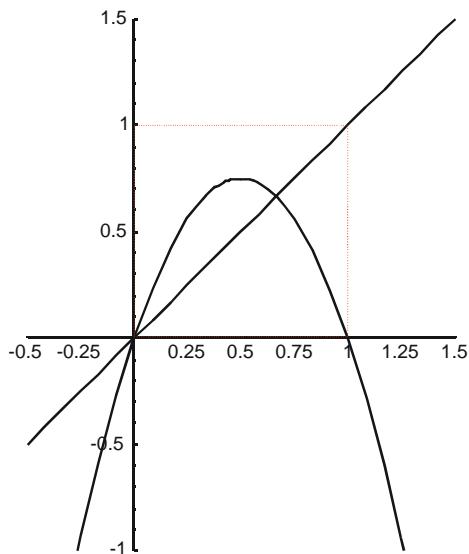
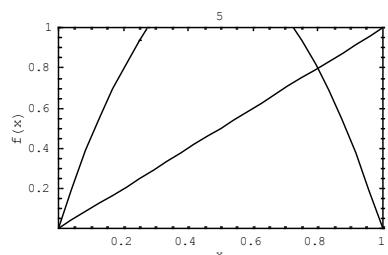
$1 < r < 4$



$r=4$



$r > 4$

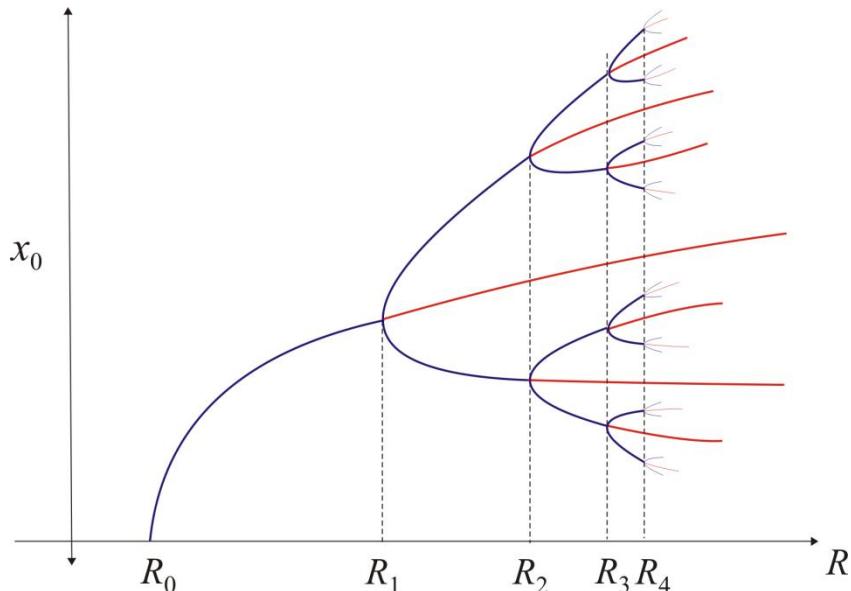


- $f(x)$ has always a maximum at $x=1/2$
- The interval $[0,1]$ is invariant under f for $r \leq 4$
- For $x_0 < 0$ $\dot{\eta}$ $x_0 > 1 \Rightarrow \lim_{n \rightarrow \infty} x_n = -\infty$
- For $r > 4$ there exist open subsets $S \subset (0,1)$ such that if $x_0 \in S \Rightarrow \lim_{n \rightarrow \infty} x_n = -\infty$

[[fp/po study](#)]

Bifurcations & period doubling

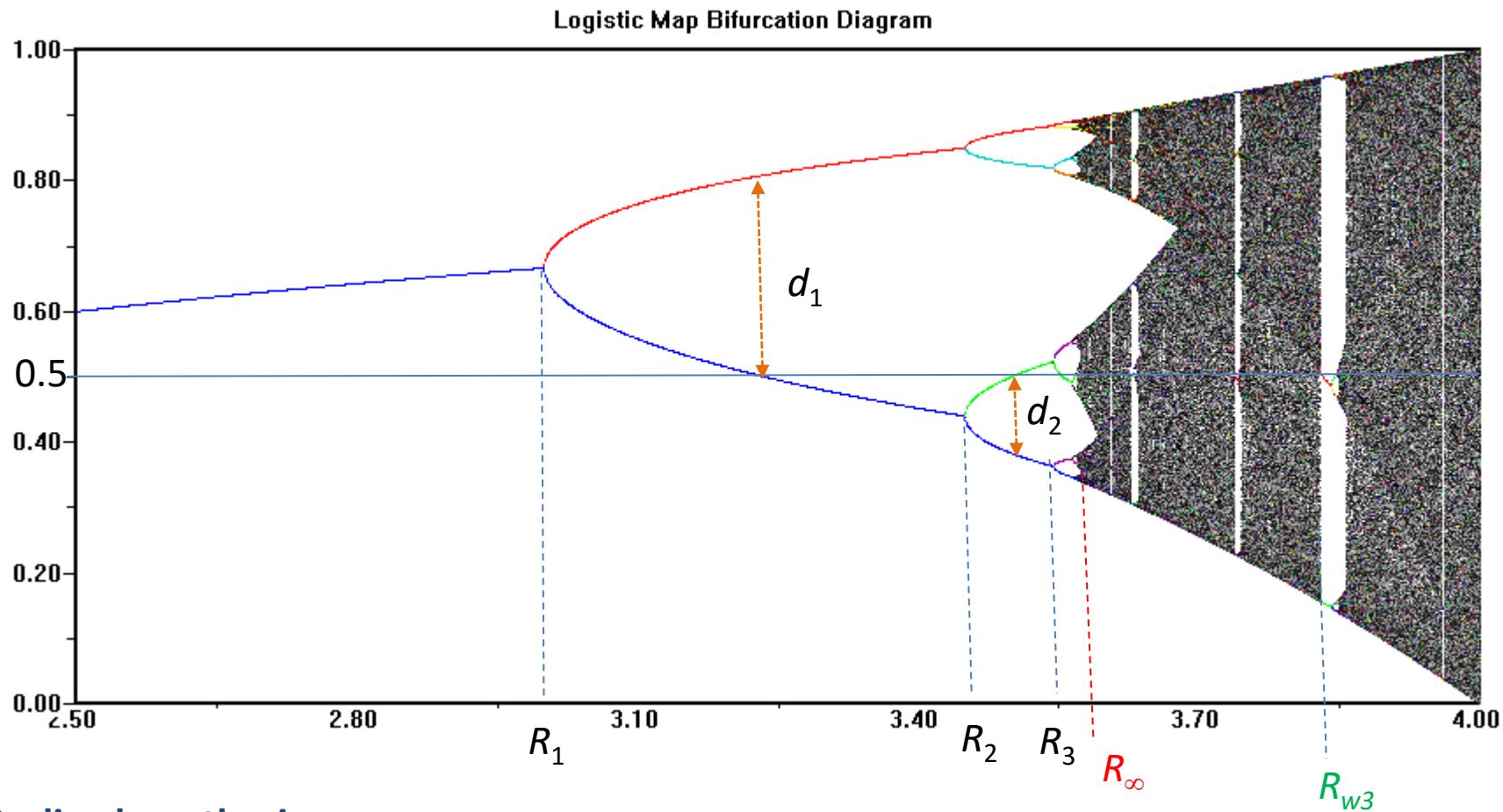
		stable	unstable	R
Fixed point	$x^*=0$	$0 < r < 1$	$r > 1$	
Fixed point	$x^* = \frac{r-1}{r}$	$1 < r < R_1$	$r > R_1$	$R_1 = 3$
Period-2	$x_{1,2}^* = \frac{1 + r \pm \sqrt{(r-3)(r+1)}}{2r}$	$R_1 < r < R_2$	$r > R_2$	$R_2 = 1 + \sqrt{6} \approx 3.4495$
Period-4	numerical	$R_2 < r < R_3$	$r > R_3$	$R_3 \approx 3.54409$
Period-8



Bifurcation diagram of stable branches

[[bifurcation nb](#)]

[[bifurcation exe](#)]



Scaling hypothesis

$$R_\infty - R_n \approx \text{const} \cdot \delta^{-n} \quad (n \rightarrow \infty)$$

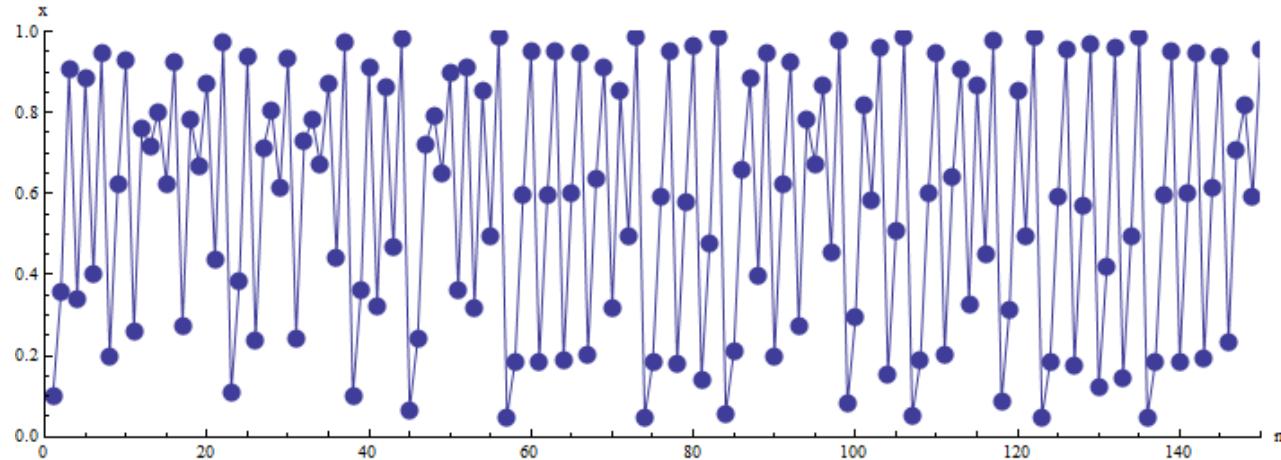
$$\delta \approx \frac{R_n - R_{n-1}}{R_{n+1} - R_n} \approx 4.6692016$$

$$a \approx \frac{d_n}{d_{n+1}} \approx 2.50290 \quad (n \rightarrow \infty)$$

$$R_\infty \approx \frac{R_n R_{n+2} - R_{n+1}^2}{R_n - 2R_{n+1} + R_{n+2}} \approx 3.5699456$$

Chaos

$$r \in (R_\infty, 4) - \bigcup \Delta_i, \quad \Delta_i = (R_i, R'_i)$$



A. Transitivity. The map f is transitive in an invariant set S if

$$\forall U \in S, V \in S, U \cap V = \emptyset \Rightarrow \exists n \text{ s.t. } U \cap f^n(V) \neq \emptyset$$

If $U \cap f^n(V) \neq \emptyset \quad \forall n > n_0 \Rightarrow \text{mixing}$

B. Sensitivity in initial conditions. $|x_0 - x'_0| < \varepsilon < \delta \Rightarrow \exists n > 0 \quad |x_n - x'_n| > \delta$

C. Periodic orbits. Periodic orbits form a dense set in S

Lyapunov exponent



$$x_N = f^N(x_0) \quad , \quad x'_N = f^N(x_0 + \varepsilon) \quad , \quad \lambda = \lambda(x_0), \quad \varepsilon \ll 1$$

$$\varepsilon e^{\lambda N} = |f^N(x_0 + \varepsilon) - f^N(x_0)| \Rightarrow \lambda = \frac{1}{N} \log \left| \frac{f^N(x_0 + \varepsilon) - f^N(x_0)}{\varepsilon} \right|$$

For $\varepsilon \rightarrow 0$ and $N \rightarrow \infty$

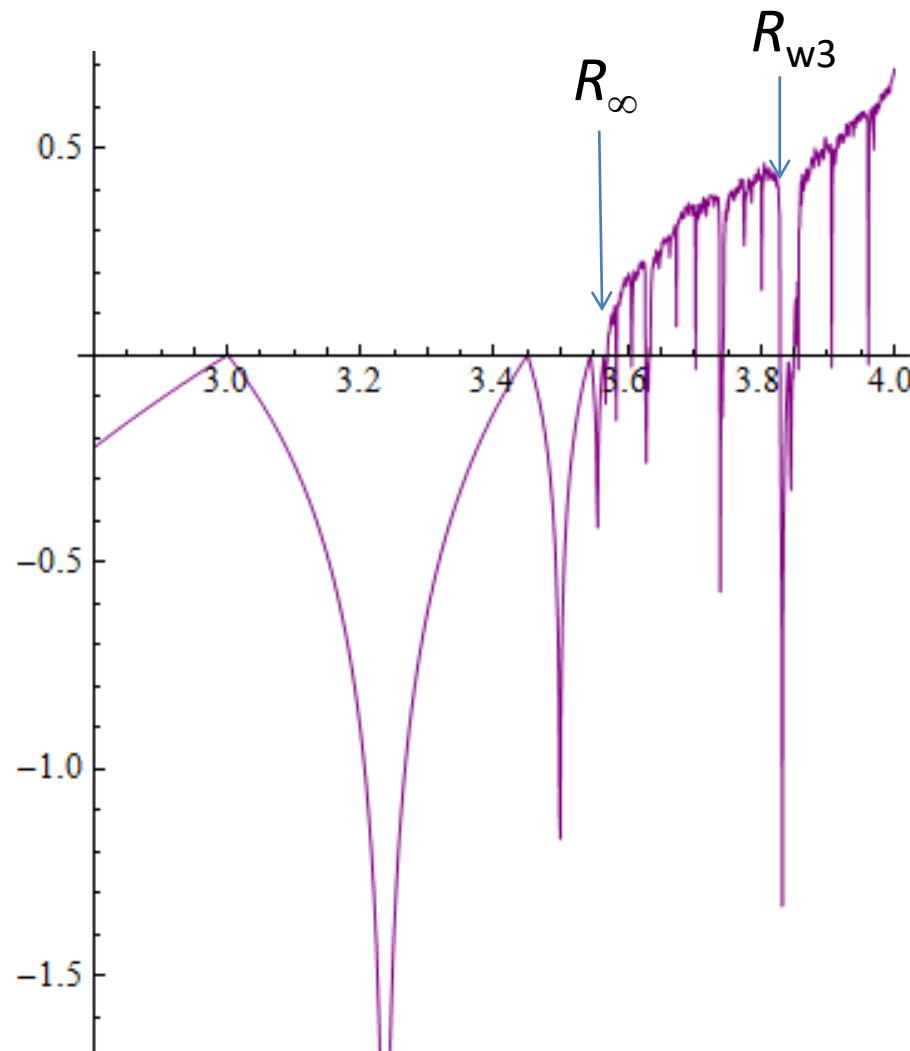
$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left| \left(\frac{df^N(x)}{dx} \right)_{x=x_0} \right| \Rightarrow \lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left| \prod_{i=0}^{N-1} f'(x_i) \right| \Rightarrow$$

$$\lambda(x_0) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \log |f'(x_i)|$$

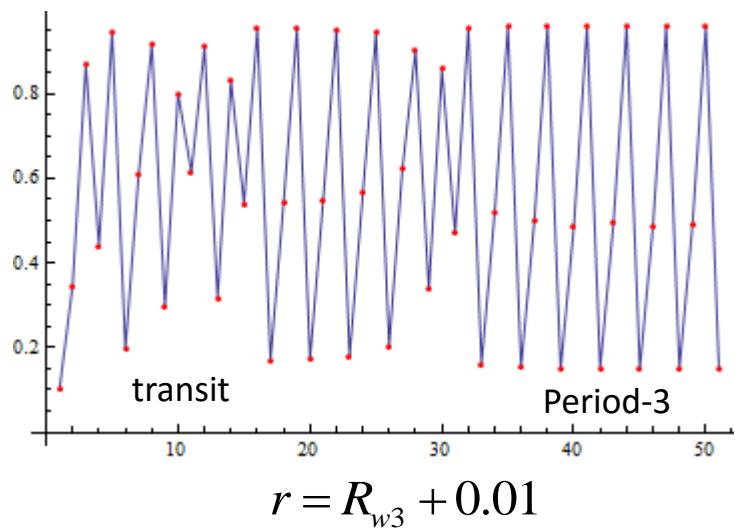
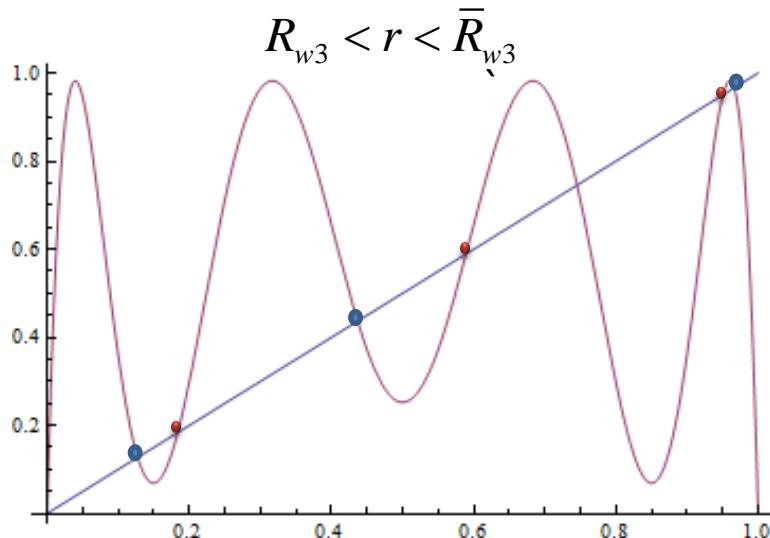
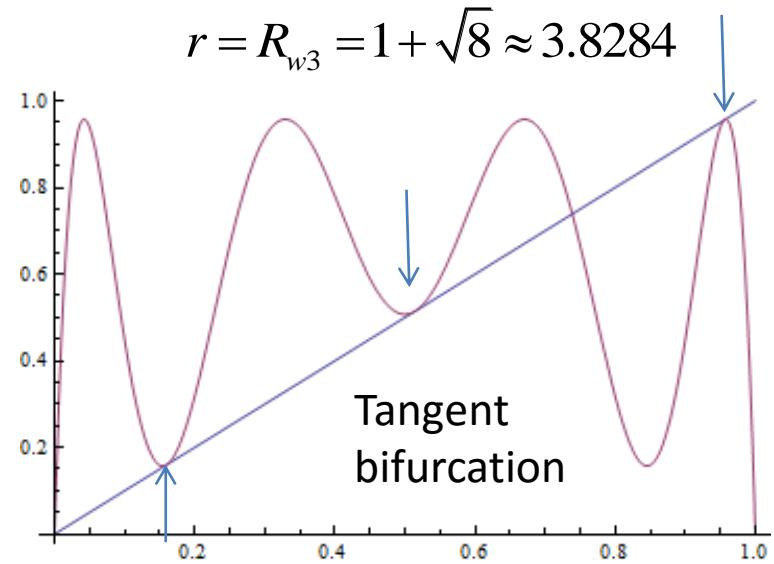
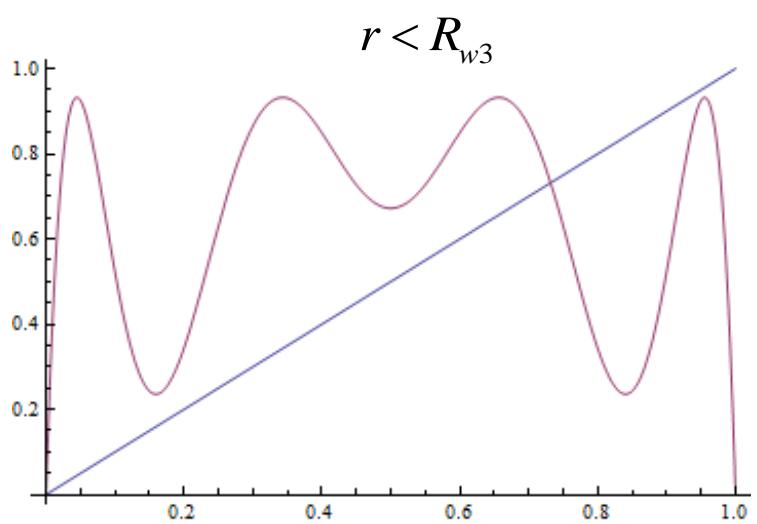
$\lambda < 0$: shrinking \rightarrow attraction
 $\lambda > 0$: expanding \rightarrow chaos

[[LCN](#)] [[LCN diagram](#)]

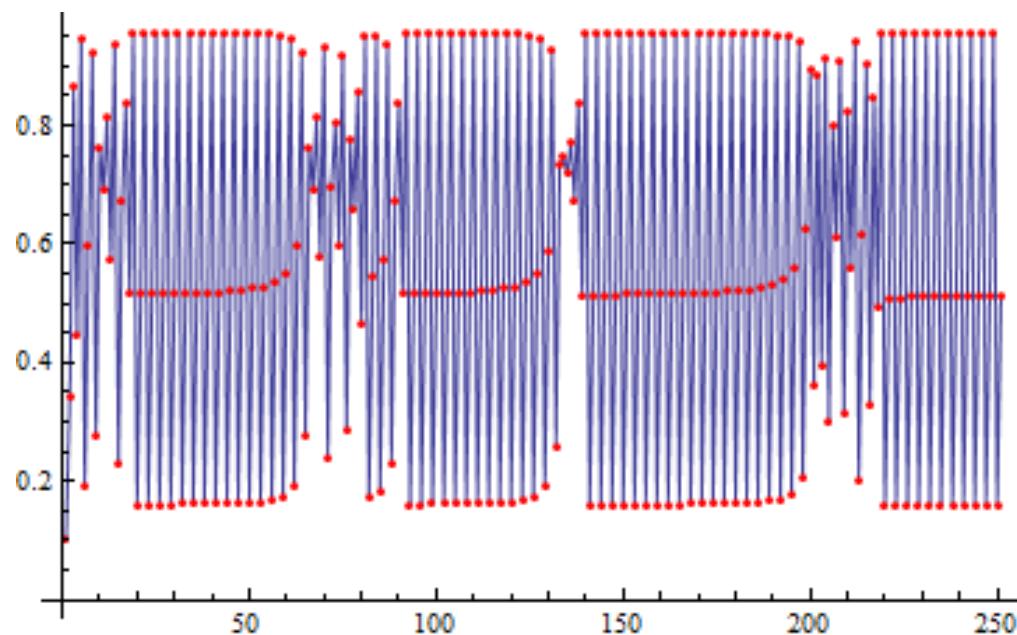
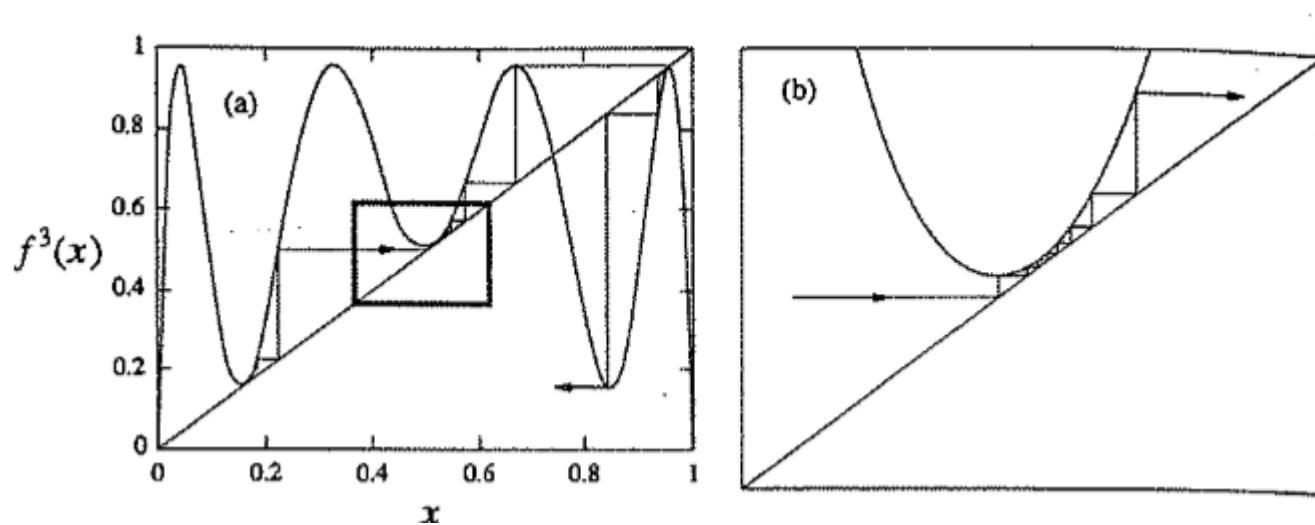
Lyapunov exponent for logistic map



The 3-period window



Tangent bifurcations & intermittency



Universal behavior of quadratic maps

A continuously differentiable map f which

- 1) maps the interval $[a,b]$ to itself
- 2) It has a single maximum in $[a,b]$
- 3) The Schwartzian derivative is negative in whole interval $[a,b]$

displays an infinite sequence of pitchfork bifurcations with the same constants α and δ .

Examples

$$x_{n+1} = r + x_n - x_n^3$$

$$x_{n+1} = rx_n e^{-x_n}$$

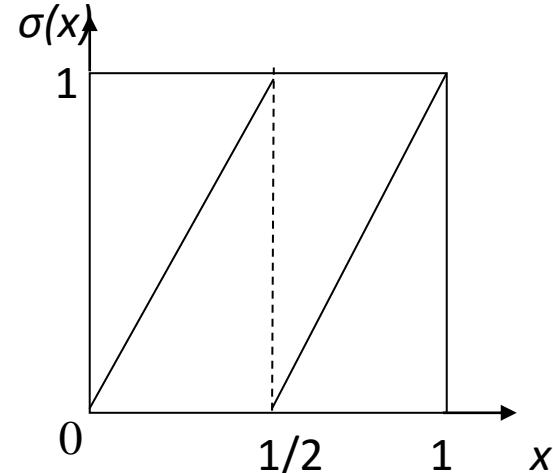
$$x_{n+1} = r \cos(\pi x_n)$$

Piecewise linear maps

The Bernoulli Shift and Deterministic Chaos

$$x_{n+1} = \sigma(x_n) = 2x_n \bmod 1, \quad x_n \in [0,1)$$

$$x = a_0, a_1 a_2 a_3 \dots \dots a_N a_{N+1} \dots \dots , \quad a_i \in \{0,1\}$$



a) the operation (mod 1)

$$x = a_0, a_1 a_2 a_3 \dots a_N a_{N+1} \dots \rightarrow x = 0, a_1 a_2 a_3 \dots a_N a_{N+1} \dots$$

b) the operation ×2

$$x = a_0, a_1 a_2 a_3 \dots a_N a_{N+1} \dots \rightarrow x = a_0 a_1, a_2 a_3 \dots a_N a_{N+1} \dots$$

c) Complete operation

$$x_1 = \sigma(x_0) = \sigma(a, a_1 a_2 a_3 \dots) = 0, a_2 a_3 a_4 \dots$$

$$x_n = \sigma^n(x_0) = \sigma^n(a, a_1 a_2 a_3 \dots) = 0, a_{n+1} a_{n+2} \dots$$

Piecewise linear maps

The Bernoulli Shift and Deterministic Chaos

Suppose the orbits $x_n = \sigma^n(x_0)$, $y_n = \sigma^n(y_0)$

$$x_0 = 0, a_1 a_2 \dots a_\nu b_{\nu+1} b_{\nu+2} \dots \quad y_0 = 0, a_1 a_2 \dots a_\nu c_{\nu+1} c_{\nu+2} \dots \quad |x_0 - y_0| = O(2^{-(\nu+1)})$$

$$x_1 = 0, a_2 a_3 \dots a_\nu b_{\nu+1} b_{\nu+2} \dots \quad y_1 = 0, a_2 a_3 \dots a_\nu c_{\nu+1} c_{\nu+2} \dots \quad |x_1 - y_1| = O(2^{-\nu})$$

$$x_2 = 0, a_3 a_4 \dots a_\nu b_{\nu+1} b_{\nu+2} \dots \quad y_2 = 0, a_3 a_4 \dots a_\nu c_{\nu+1} c_{\nu+2} \dots \quad |x_2 - y_2| = O(2^{-(\nu-1)})$$

.....

$$x_\nu = 0, b_{\nu+1} b_{\nu+2} \dots \quad y_0 = 0, c_{\nu+1} c_{\nu+2} \dots \quad |x_\nu - y_\nu| = O(2^{-1})$$

Exponential divergence  chaos

B) 2D discrete maps

2D discrete maps

$$\mathbf{G} : \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

$$\mathbf{z}_{n+1} = \mathbf{G}(\mathbf{z}_n), \quad \mathbf{z} = (x, y)$$

$$x_{k+1} = g_1(x_k, y_k)$$

$$y_{k+1} = g_2(x_k, y_k)$$

Orbit = { $P_0, P_1, P_2, P_3, \dots$ }

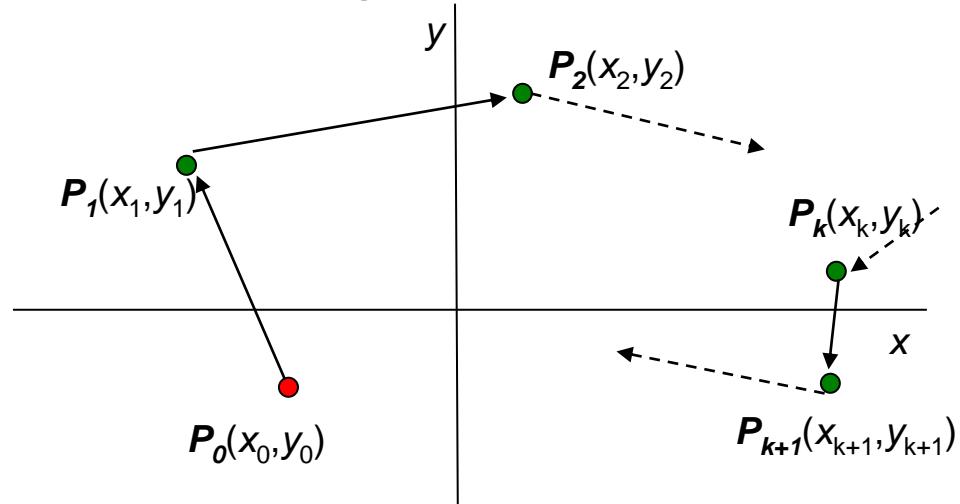
$$P_k = G^k(P_0), \quad k = 1, 2, \dots$$

Invertibility: there exist \mathbf{G}^{-1}

$$\begin{aligned} x_{k-1} &= \bar{g}_1(x_k, y_k) & , \quad \mathbf{G}^{-1} = (\bar{g}_1, \bar{g}_2) \\ y_{k-1} &= \bar{g}_2(x_k, y_k) \end{aligned}$$

Orbit = { $\dots, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \dots$ }

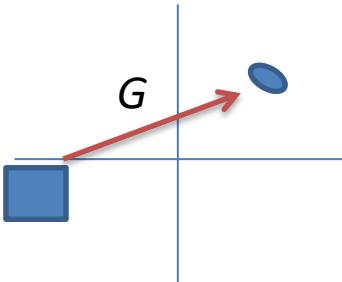
A set S is **invariant** under \mathbf{G} if $\forall \mathbf{z} \in S \Rightarrow \mathbf{G}(\mathbf{z}) \in S$



Area preservation

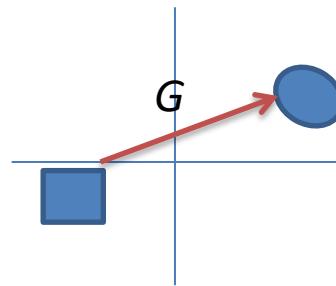
The Jacobian

$$\mathbf{M} = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial x_n} & \frac{\partial g_1}{\partial y_n} \\ \frac{\partial g_2}{\partial x_n} & \frac{\partial g_2}{\partial y_n} \end{pmatrix}$$



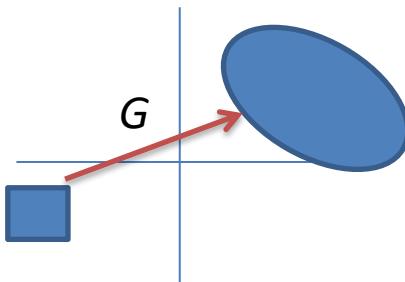
$$|\det \mathbf{M}| < 1$$

Dissipative



$$|\det \mathbf{M}| = 1$$

Area preserving
(conservative)



$$|\det \mathbf{M}| > 1$$

Exploding

Special solutions

- Fixed points (equilibriums)

$$P^* = \mathbf{G}(P^*) \quad \begin{cases} x^* = g_1(x^*, y^*) \\ y^* = g_2(x^*, y^*) \end{cases}$$

- Periodic orbits of period k

$$P^* = \mathbf{G}^k(P^*) \quad \begin{cases} x^* = g_1(\dots(g_1(g_1(x^*, y^*), g_2(x^*, y^*)), \dots) \\ y^* = g_2(\dots(g_2(g_1(x^*, y^*), g_2(x^*, y^*)), \dots) \end{cases}$$

Linear stability

adjacent linear map

$$P^* = (x^*, y^*)$$

$$\begin{pmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = \mathbf{M}^* \begin{pmatrix} \Delta x_k \\ \Delta y_k \end{pmatrix}, \quad \mathbf{M}^* = \begin{pmatrix} \frac{\partial g_1}{\partial x_n} & \frac{\partial g_1}{\partial y_n} \\ \frac{\partial g_2}{\partial x_n} & \frac{\partial g_2}{\partial y_n} \end{pmatrix}_{(x^*, y^*)}$$

λ_1, λ_2 : eigenvalues

- If $|\lambda_1| > 1$ and /or $|\lambda_2| > 1$ then P^* is **unstable**
- If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then P^* is **asymptotically stable**
- If λ_2, λ_1 are complex conjugate with $|\lambda_1| = |\lambda_2| = 1$ then P^* is a **center**

* Correspondingly for k -periodic orbits using the Jacobian of \mathbf{G}^k .

Example: the dissipative Henon map

$$\mathbf{G} : \begin{aligned}x_{n+1} &= 1 - ax_n^2 + y_n \\y_{n+1} &= bx_n\end{aligned}\quad \det M = b \quad (0 < b < 1)$$

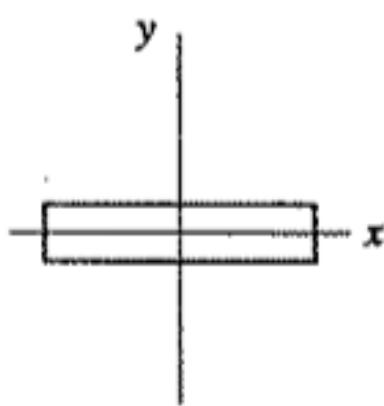
$$\mathbf{G} = \mathbf{G}_3 \circ \mathbf{G}_2 \circ \mathbf{G}_1$$

$$\mathbf{G}_1 : \begin{aligned}x_{n+1} &= x_n \\y_{n+1} &= 1 - ax_n^2 + y_n\end{aligned}\quad \mathbf{G}_2 : \begin{aligned}x_{n+2} &= bx_{n+1} = bx_n \\y_{n+2} &= y_{n+1} = 1 - ax_n^2 + y_n\end{aligned}\quad \mathbf{G}_3 : \begin{aligned}x_{n+3} &= y_{n+2} = 1 - ax_n^2 + y_n \\y_{n+3} &= x_{n+2} = bx_n\end{aligned}$$

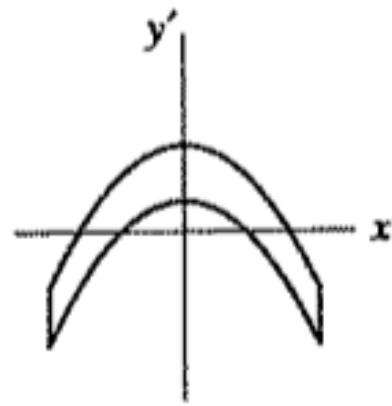
(area preserving bending)

(Contraction in the x-direction)

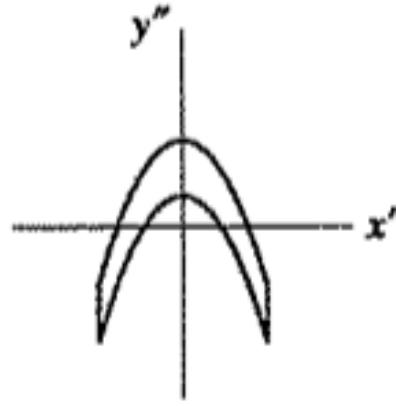
(reflection)



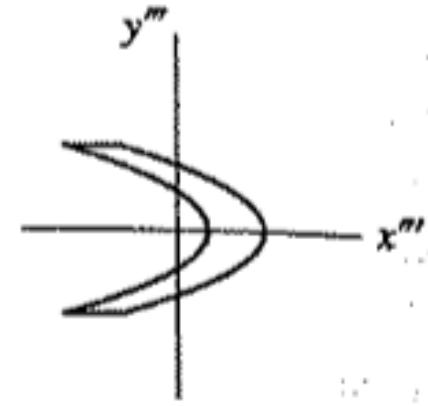
(a)



(b)



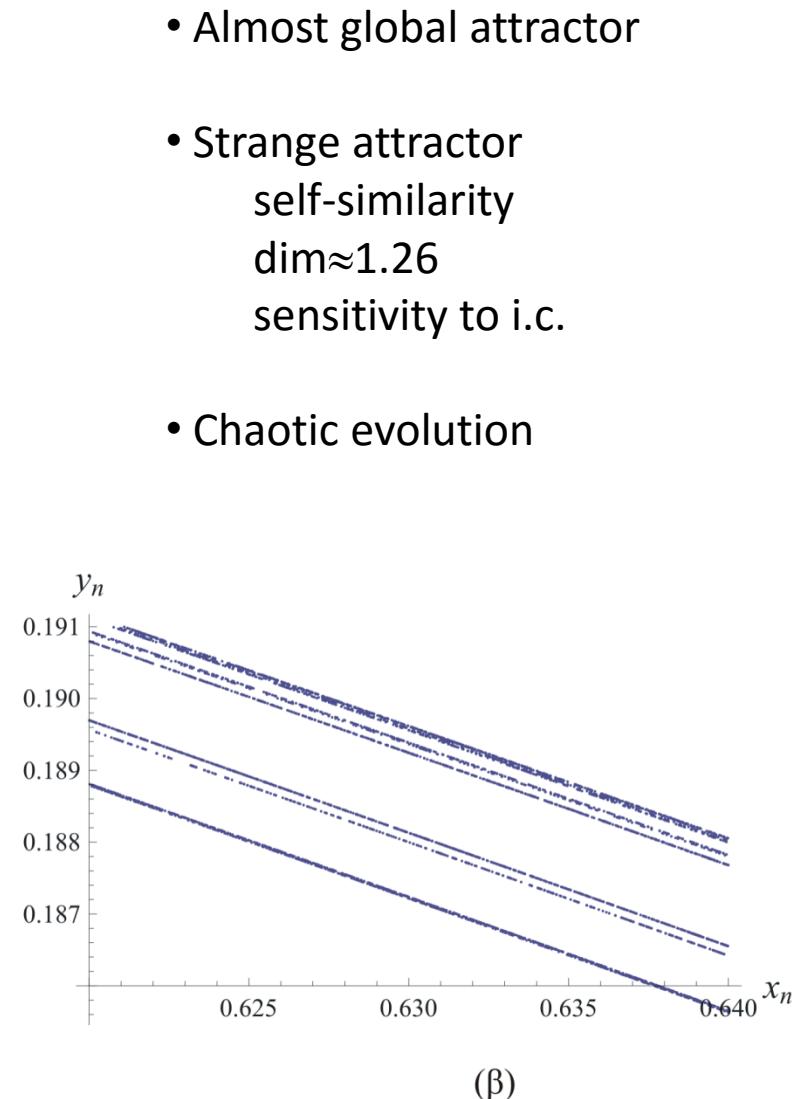
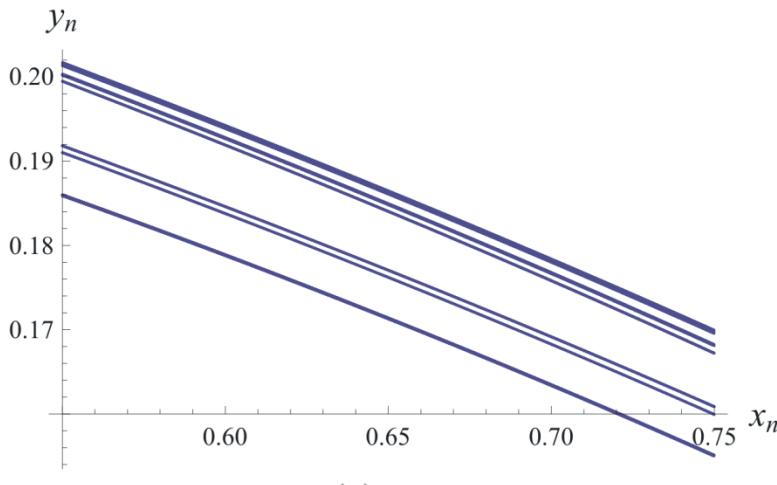
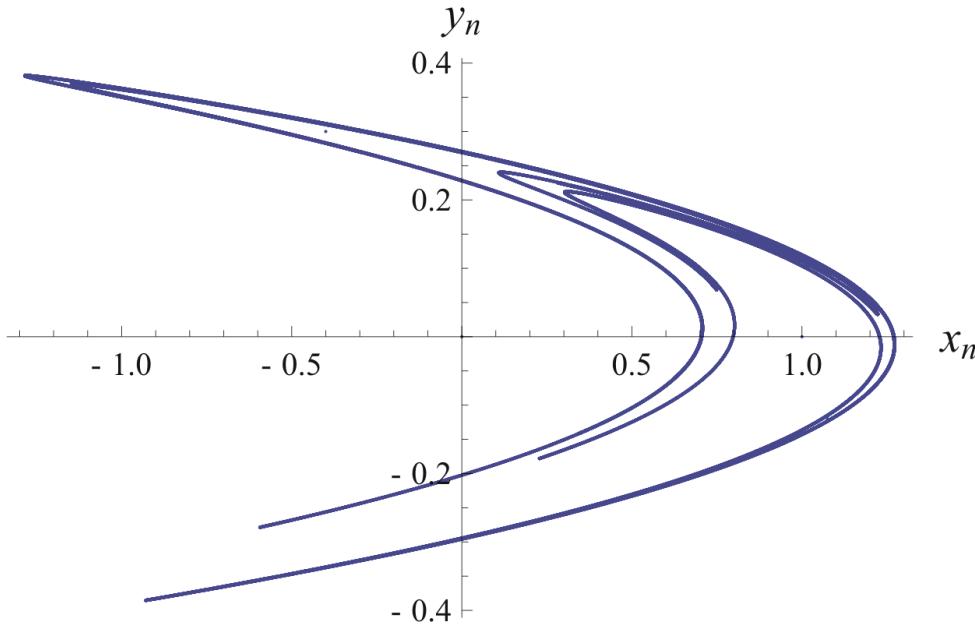
(c)



(d)

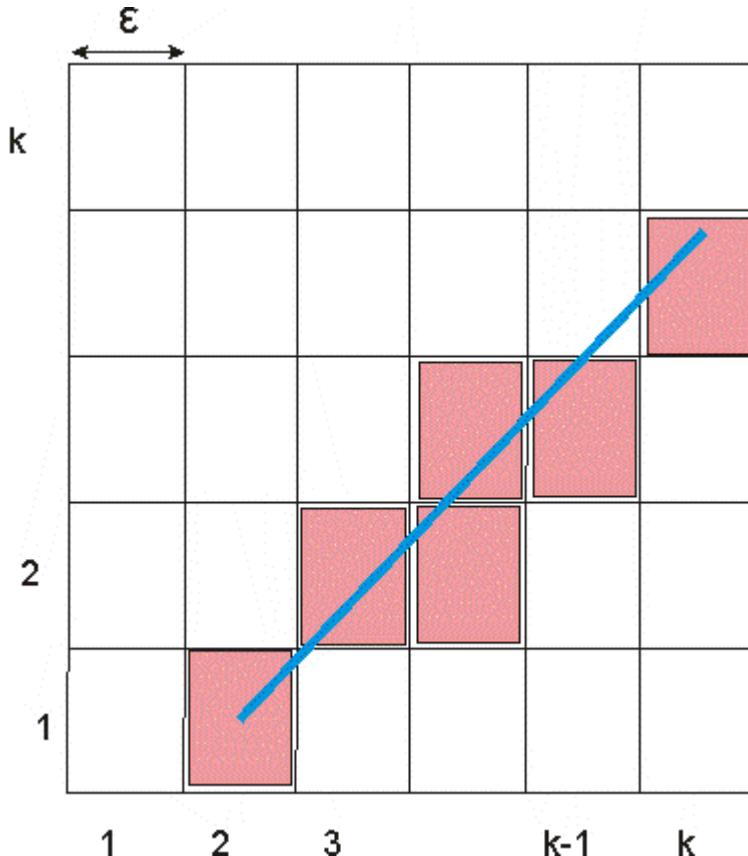
[[Henon](#)]

Henon attractor



- Almost global attractor
- Strange attractor
self-similarity
 $\dim \approx 1.26$
sensitivity to i.c.
- Chaotic evolution

Box (Hausdorff) dimension



k^D : number of boxes (space of dim= D)

ε : length of the box

$$N(\varepsilon) = \frac{\text{const}}{\varepsilon}$$

Number of boxes covered
by a curve (dim=1)

$$N(\varepsilon) = \frac{\text{const}}{\varepsilon^2}$$

Number of boxes covered
by a surface (dim=2)

$$N(\varepsilon) = \frac{\text{const}}{\varepsilon^d}$$

Number of boxes covered
by an object of dim= $d \in \mathbb{R}$

$$d = -\frac{\ln N(\varepsilon) - \ln(\text{const})}{\ln \varepsilon} \quad \rightarrow$$

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$$

[\[H-box dim\]](#)

Area preserving or
Conservative systems

Stability in conservative maps

Det $|\mathbf{M}|=1 \Rightarrow (\lambda_1, \lambda_2)$ form a reciprocal pair ($\lambda_1\lambda_2=1$)

(i) $\lambda_1, \lambda_2 \in R, \quad \lambda_2 = \frac{1}{\lambda_1}$

(ii) $\lambda_1, \lambda_2 \in C \setminus R, \quad \lambda_1 = e^{i\varphi}, \lambda_2 = e^{-i\varphi}$

Stability index k

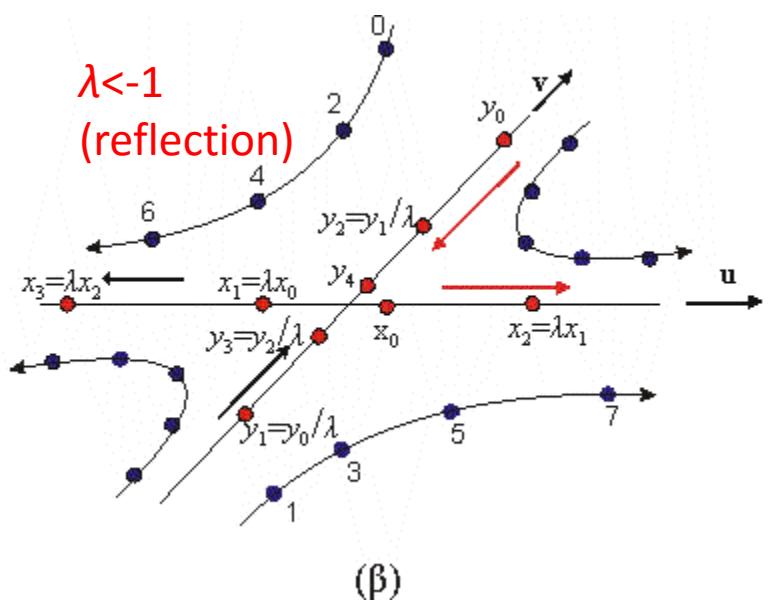
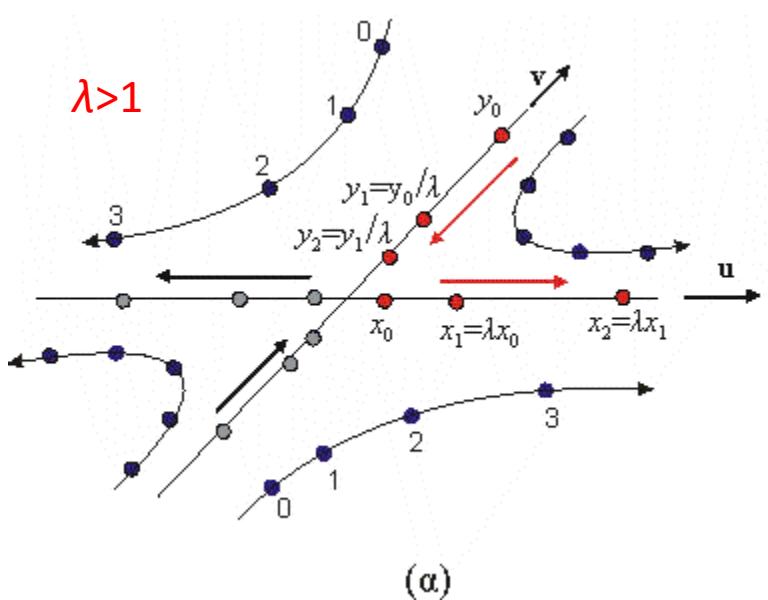
$$\lambda_1, \lambda_2 = \frac{1}{2} \left(T \pm \sqrt{T^2 - 4} \right), \quad T = \text{Trace}\mathbf{M}(x^*, y^*)$$

$$k = |\text{Trace}\mathbf{M}(x^*, y^*)| < 2 \quad \text{case (i)}$$

$$k = |\text{Trace}\mathbf{M}(x^*, y^*)| > 2 \quad \text{case (ii)}$$

Stability in conservative maps

- $\lambda_1, \lambda_2 \in R$, $\lambda_2 = \frac{1}{\lambda_1} = \lambda$, $|\lambda| > 1$
- $$\Delta x_{k+1} = \lambda \Delta x_k \quad , \quad \Delta y_{k+1} = \frac{1}{\lambda} \Delta y_k$$
- | | | |
|-----------------|---|--|
| $(\lambda > 1)$ | $\Delta x_0 > 0, \lim_{k \rightarrow \infty} \Delta x_k = \infty$ | $(\Delta x_0 < 0, \lim_{k \rightarrow \infty} \Delta x_k = -\infty)$ |
| | $\Delta y_0 > 0, \lim_{k \rightarrow \infty} \Delta y_k = 0$ | $(\Delta y_0 < 0, \lim_{k \rightarrow \infty} \Delta y_k = 0)$ |

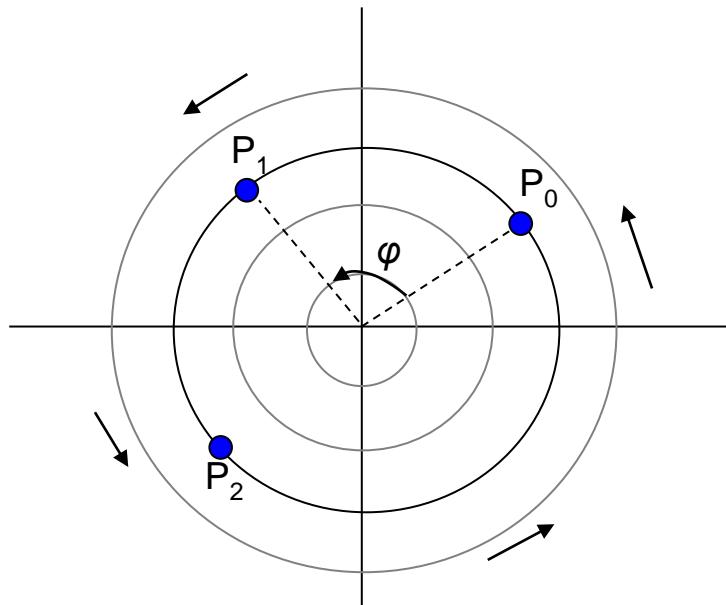


Hyperbolic point (unstable)

Stability in conservative maps

- $\lambda_1, \lambda_2 \in C \setminus R, \quad \lambda_1 = e^{i\varphi}, \quad \lambda_2 = e^{-i\varphi} \quad (|\lambda_i| = 1)$

$$\begin{pmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \Delta x_k \\ \Delta y_k \end{pmatrix} = \mathbf{R} \begin{pmatrix} \Delta x_k \\ \Delta y_k \end{pmatrix}$$



Elliptic point (stable)

Example: the Eanston's map

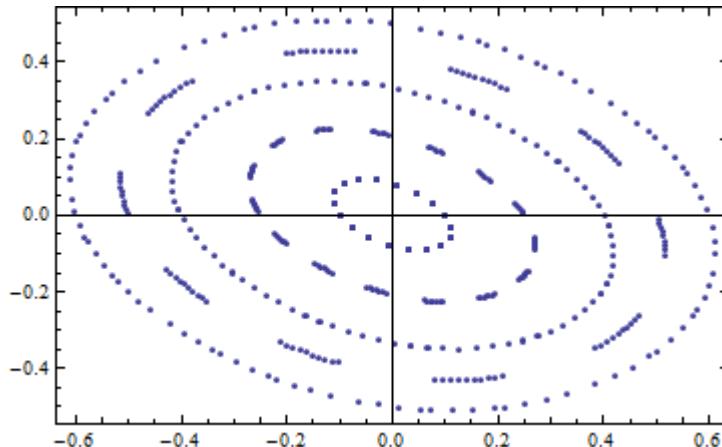
$$x_{n+1} = a(x_n \cos \theta_n + y_n \sin \theta_n)$$

$$y_{n+1} = \frac{1}{a}(-x_n \sin \theta_n + y_n \cos \theta_n)$$

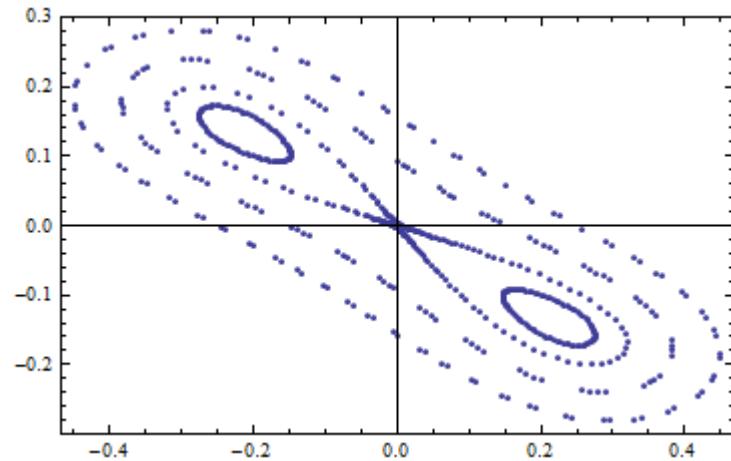
$$\theta_n = (x_n^2 + y_n^2 + \varphi)^b$$

$$a, b, \varphi = \text{const. } > 0$$

$P^*=(0,0)$ fixed point, stability criterion : $a < \frac{1 + \sin \varphi^b}{\cos \varphi^b}$

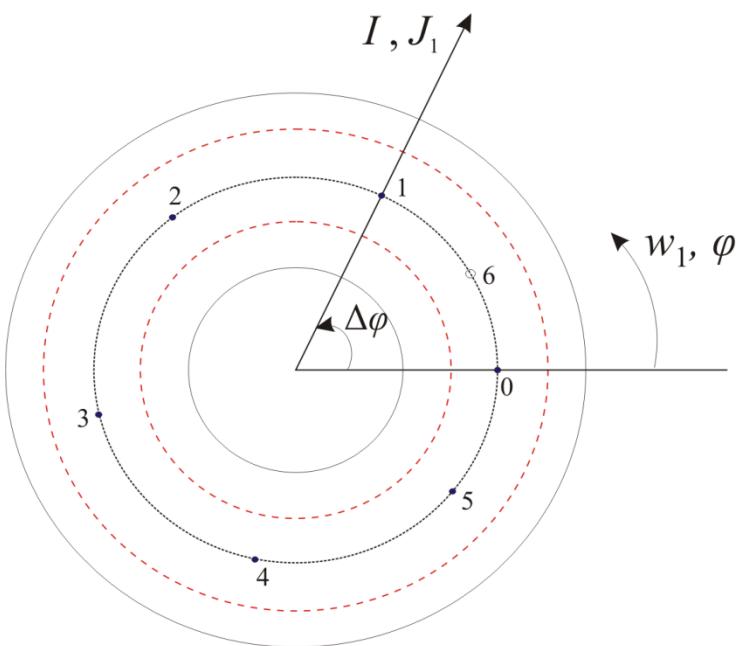
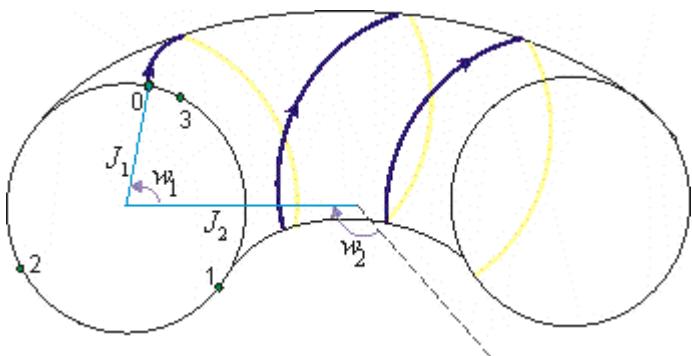


$$a=1.2, b=1.5, \varphi=\pi/6$$



$$a=1.6, b=1.5, \varphi=\pi/6$$

The twist map

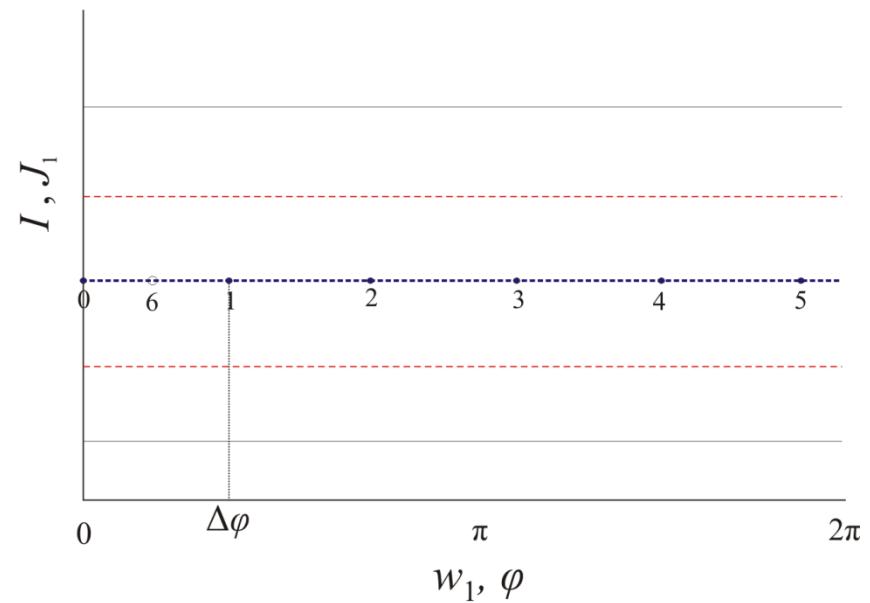


(a)

$$I_{n+1} = I_n$$

$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n)$$

Periodicity condition : $\rho(I^*) = \frac{p}{q} \in \mathbb{Q}$



(b)

The perturbed twist map

$$I_{n+1} = I_n + \varepsilon f(I_n, \varphi_n)$$

$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n) + \varepsilon g(I_n, \varphi_n)$$

$$\det |M| = 1$$

$$\varepsilon \ll 1$$

The perturbed twist map

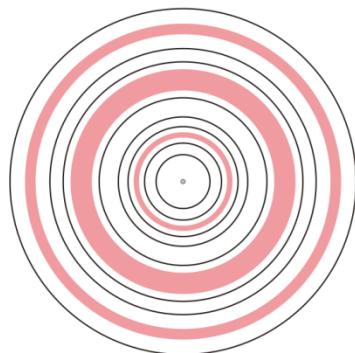
$$I_{n+1} = I_n + \varepsilon f(I_n, \varphi_n)$$

$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n) + \varepsilon g(I_n, \varphi_n) \quad \varepsilon \ll 1$$

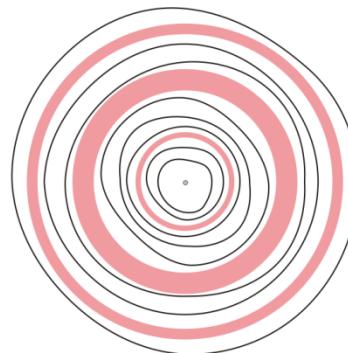
KAM theorem

Preservation of invariant curves far from resonances

$$\left| \rho(I) - \frac{p}{q} \right| > \frac{K(\varepsilon)}{q^{5/2}} \quad \forall p, q \in N$$



$$\varepsilon=0$$



$$0 < \varepsilon \ll 1$$

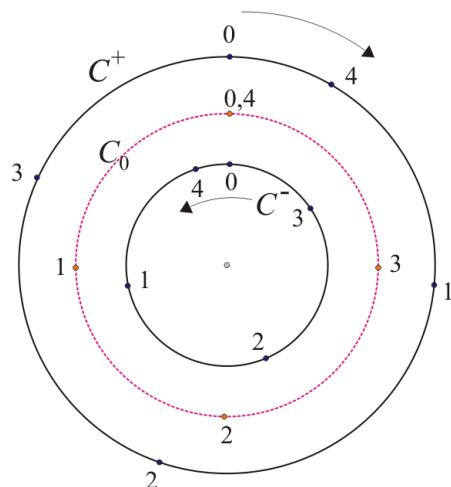
The perturbed twist map

$$I_{n+1} = I_n + \varepsilon f(I_n, \varphi_n)$$

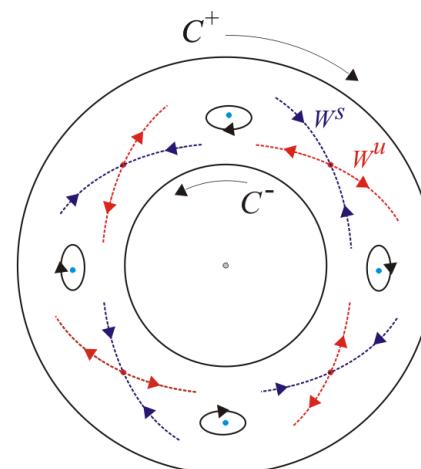
$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n) + \varepsilon g(I_n, \varphi_n) \quad \varepsilon \ll 1$$

Poincare – Birkhoff theorem

For each resonant invariant circle only an **even number** from the infinite number of **periodic orbits** survive under the perturbation



$$\varepsilon=0$$

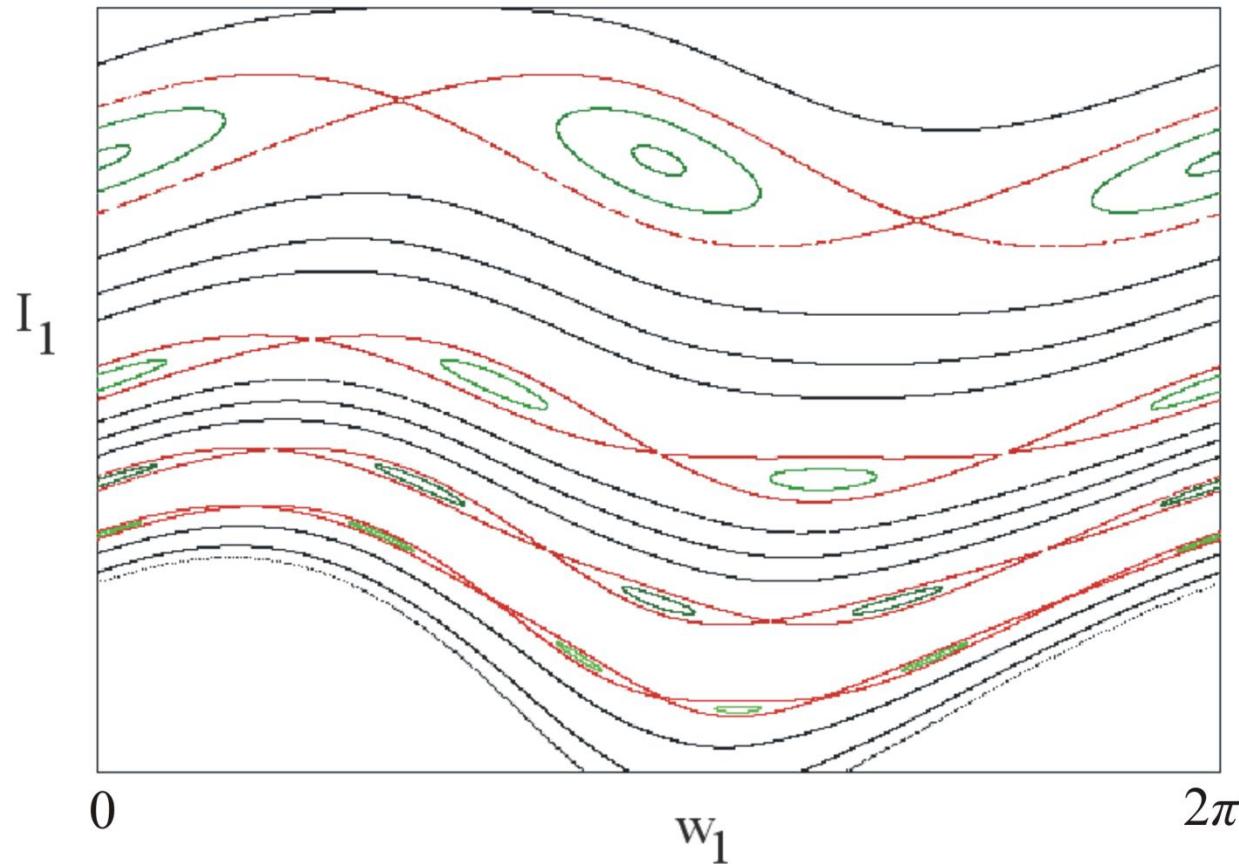


$$0 < \varepsilon \ll 1$$

The perturbed twist map

$$I_{n+1} = I_n + \varepsilon f(I_n, \varphi_n)$$

$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n) + \varepsilon g(I_n, \varphi_n) \quad \varepsilon \ll 1$$



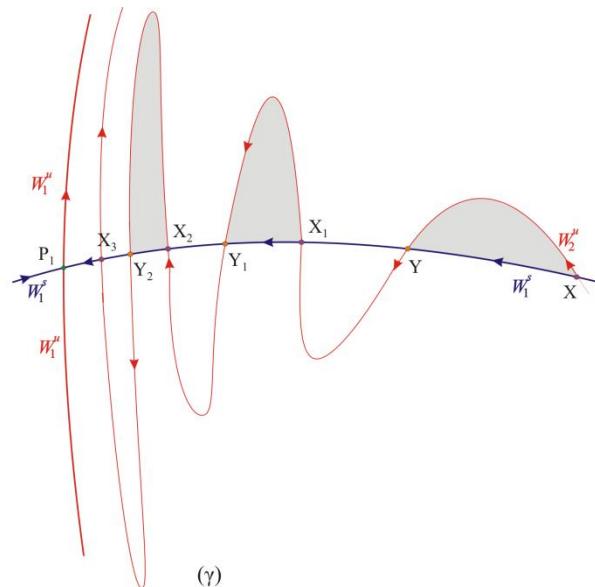
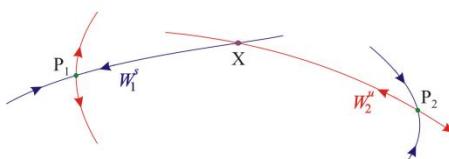
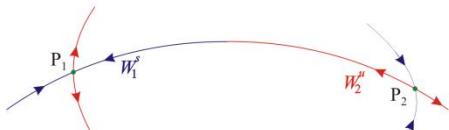
The perturbed twist map

$$I_{n+1} = I_n + \varepsilon f(I_n, \varphi_n)$$

$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n) + \varepsilon g(I_n, \varphi_n)$$

(area preserving)

Poincare – Smale theory (homoclinic chaos)

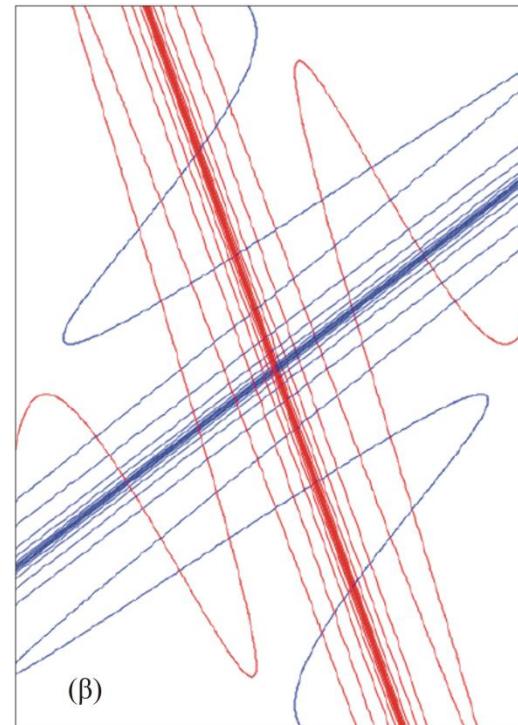
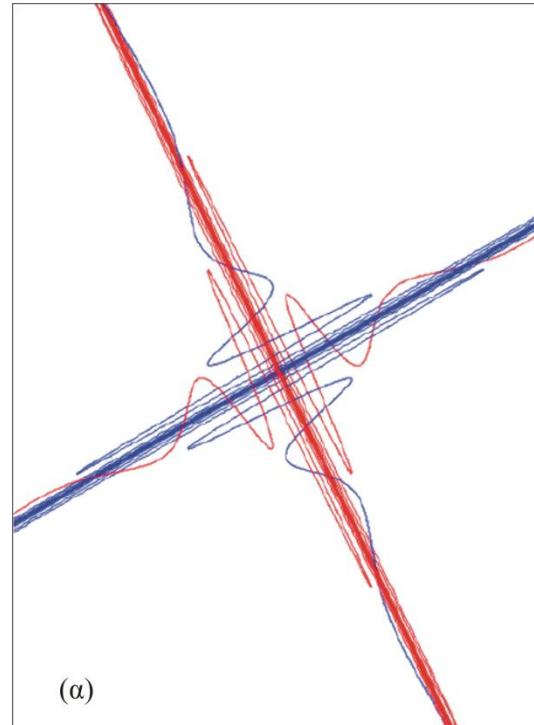


The perturbed twist map

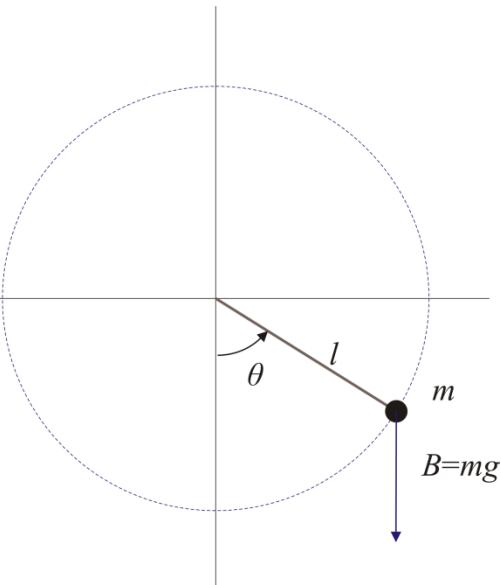
$$I_{n+1} = I_n + \varepsilon f(I_n, \varphi_n)$$

$$\varphi_{n+1} = \varphi_n + 2\pi\rho(I_n) + \varepsilon g(I_n, \varphi_n) \quad \varepsilon \ll 1$$

Poincare – Smale theory (homoclinic chaos)



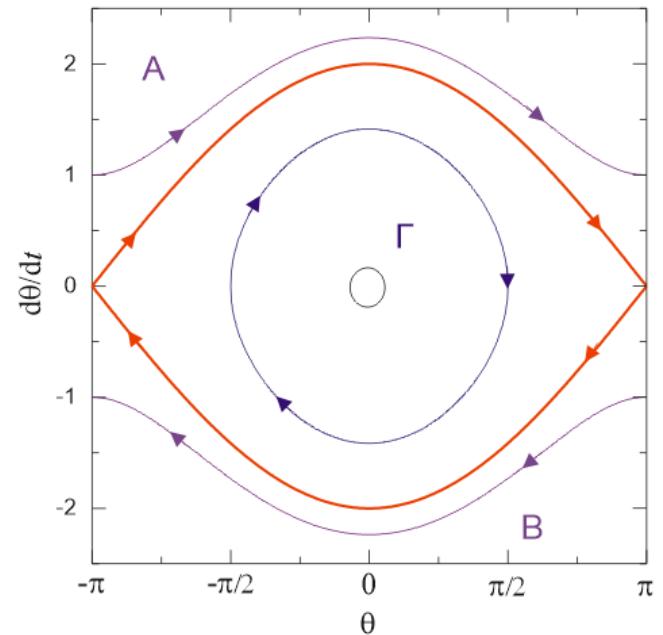
The standard map



$$\ddot{\theta} = -\omega^2 \sin \theta$$

$$\dot{\theta} = p$$

$$\dot{p} = -\omega^2 \sin \theta$$



$$\begin{aligned} \theta_n = \theta(t) &\rightarrow \theta_{n+1} = \theta(t + \Delta t) \\ p_n = p(t) &\rightarrow p_{n+1} = p(t + \Delta t) \end{aligned} \xrightarrow{\hspace{2cm}} \dot{\theta} \approx \frac{\theta_{n+1} - \theta_n}{\Delta t}, \quad \dot{p} \approx \frac{p_{n+1} - p_n}{\Delta t}$$

Substitute derivatives in ODEs at time $n\Delta t$

$$\begin{aligned} \theta_{n+1} - \theta_n &= p_n \Delta t \\ p_{n+1} - p_n &= -\omega^2 \Delta t \sin \theta_n \end{aligned} \Rightarrow \quad \begin{aligned} \theta_{n+1} - \theta_n &= I_n \\ I_{n+1} - I_n &= -k \sin \theta_n \end{aligned} \Rightarrow \quad \begin{aligned} \theta_{n+1} &= \theta_n + I_n \\ I_{n+1} &= I_n - k \sin \theta_n \end{aligned} \quad (1) \quad \text{where } I = p \Delta t \quad k = \omega^2 \Delta t^2$$

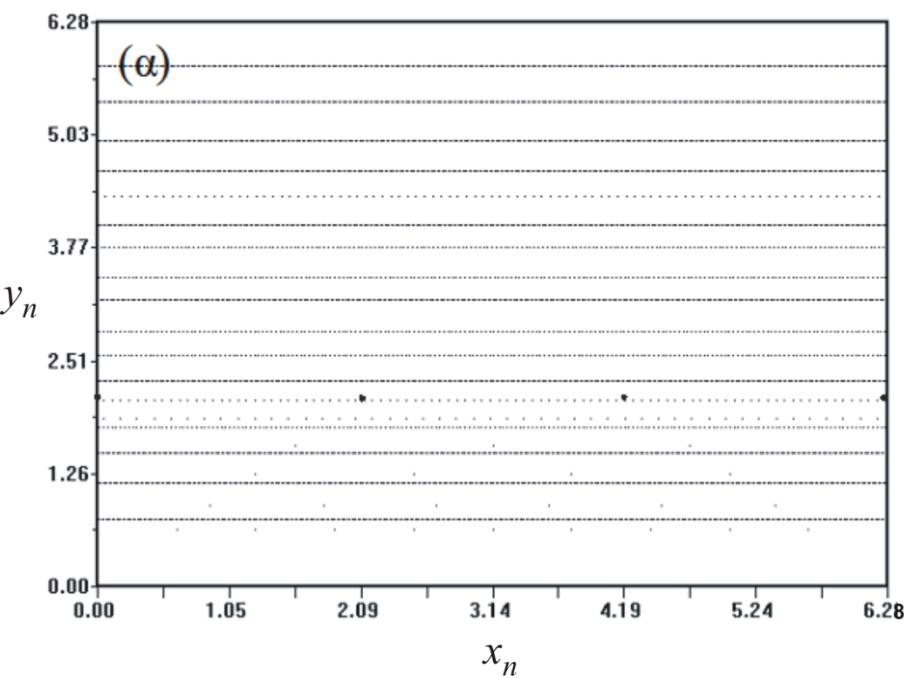
Map (1) is not conservative! It becomes conservative as

$$\boxed{\begin{aligned} \theta_{n+1} &= \theta_n + I_{n+1} \\ I_{n+1} &= I_n - k \sin \theta_n \end{aligned}}$$

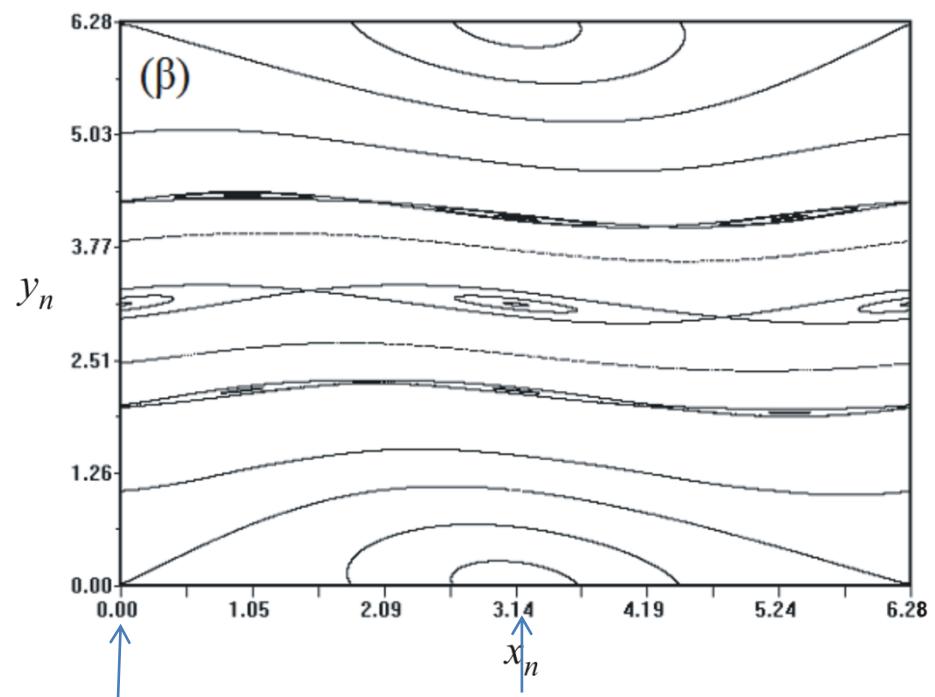
The standard map

$$\begin{aligned} I_{n+1} &= I_n + k \cos \theta_n & \theta = \text{mod } 2\pi, \quad I = \text{mod } 2\pi \\ \theta_{n+1} &= \theta_n + I_{n+1} \end{aligned}$$

$k=0$ (twist map)



$k=0.3$



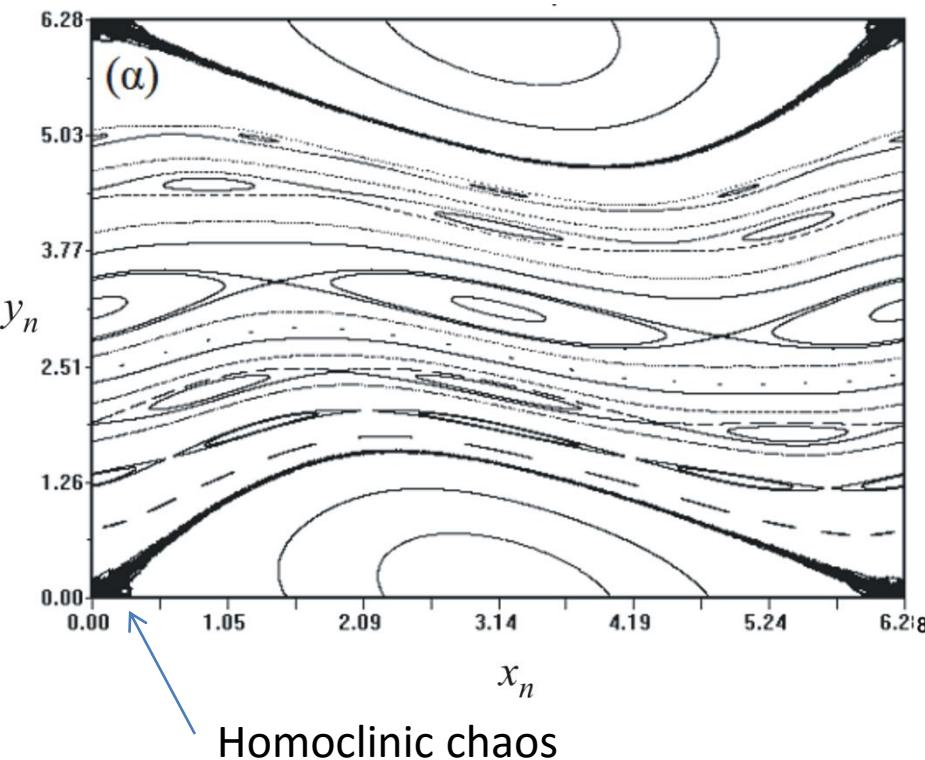
Unstable fp

Stable fp ($k<3$)

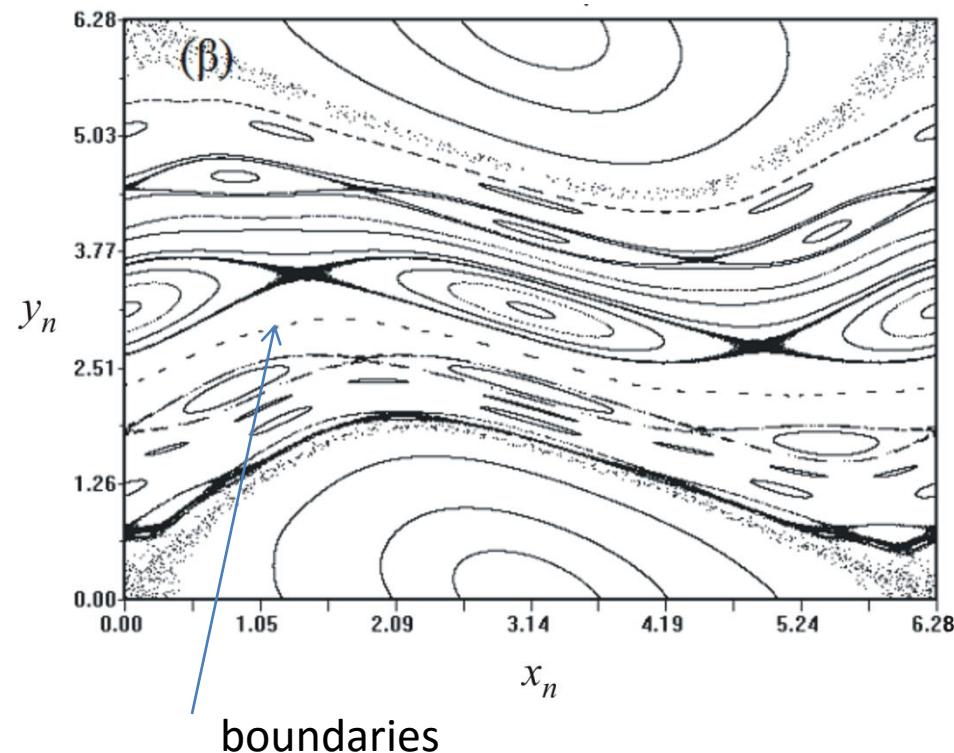
The standard map

$$\begin{aligned} I_{n+1} &= I_n + k \cos \theta_n & \theta = \text{mod } 2\pi, \quad I = \text{mod } 2\pi \\ \theta_{n+1} &= \theta_n + I_{n+1} \end{aligned}$$

$k=0.6$



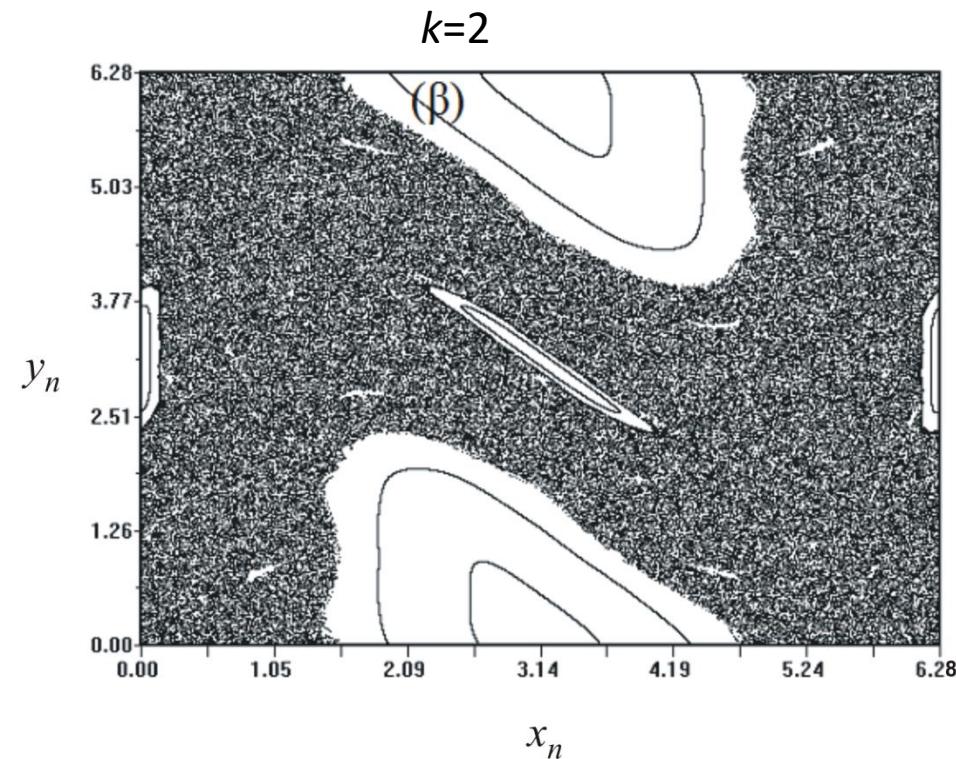
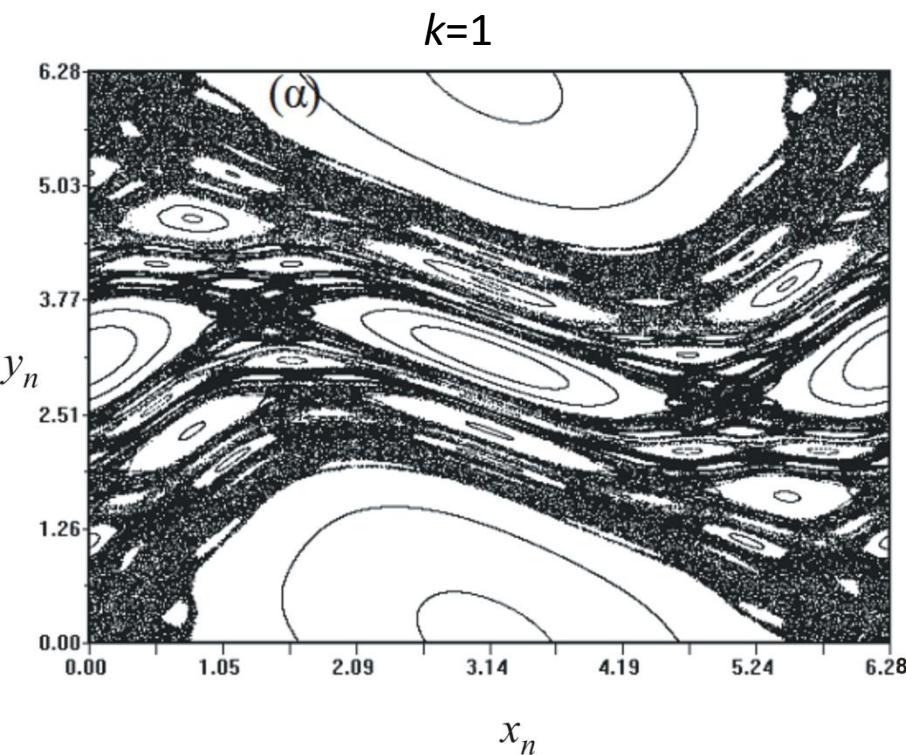
$k=0.8$



The standard map

$$\boxed{\begin{aligned} I_{n+1} &= I_n + k \cos \theta_n & \theta = \text{mod } 2\pi, \quad I = \text{mod } 2\pi \\ \theta_{n+1} &= \theta_n + I_{n+1} \end{aligned}}$$

Last boundary invariant curve $\rightarrow \rho = \frac{1+\sqrt{5}}{2}$ ($k \approx 0.98$)



Hadjidemetriou's maps

$$x_n \rightarrow x_{n+1}$$

$$y_n \rightarrow y_{n+1}$$

$$F = F(x_n, y_{n+1})$$

→

$$x_{n+1} = \frac{\partial F(x_n, y_{n+1})}{\partial y_{n+1}}, \quad y_n = \frac{\partial F(x_n, y_{n+1})}{\partial x_n}$$

A symplectic (conservative) map for any function F , which is called **generating function** of the map

- We need a suitable choice of F in order to describe a Hamiltonian system with a discrete map

e.g. Pendulum dynamics, the Hamiltonian is $H(\theta, p) = \frac{1}{2} p^2 + k \cos \theta$

We choose $F = p_{n+1}\theta_n + H(\theta_n, p_{n+1}) = p_{n+1}\theta_n + \frac{1}{2} p_{n+1}^2 + k \cos \theta_n$

$$\theta_{n+1} = \frac{\partial F}{\partial p_{n+1}} = \theta_n + p_{n+1}$$

The standard map

$$p_n = \frac{\partial F}{\partial \theta_n} = p_{n+1} - k \cos \theta_n$$

Hadjidemetriou's maps

$$H = H(I, J, \theta, \varphi) \quad \text{with } \varphi : \text{fast angle variable}$$

$$\langle H \rangle = \frac{1}{2\pi} \int_0^{2\pi} H(I, J, \theta, \varphi) d\varphi \quad \langle H \rangle = \hat{H}(I, \theta; \mathbf{J} = \text{const})$$

$$F = I_{n+1} \theta_n + 2\pi \hat{H}(\theta_n, I_{n+1})$$

$$\Delta t = 2\pi$$

Hadjidemetriou's 3:1 resonance map

$$\begin{aligned}
 S_n &= S_{n+1} + \mu T \frac{2b}{N_{n+1}} S_{n+1} \sin(2\sigma_n) \\
 &\quad - \mu e_j T \sqrt{2S_{n+1}} (G \sin(\sigma_n + v_n) + D \sin(\sigma_n - v_n)) \\
 &\quad + \mu T e_j F_1 F_2 S_{n+1}^3 \sin(\sigma_n + v_n)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{n+1} &= \sigma_n - \frac{4T\mu_1^2}{(N_{n+1} - S_{n+1})^3} + \frac{3}{2} T + 2\mu T F - \mu T \frac{b}{N_{n+1}} \cos(2\sigma_n) \\
 &\quad + \frac{\mu e_j T}{\sqrt{2S_{n+1}}} (G \cos(\sigma_n + v_n) + D \cos(\sigma_n - v_n)) \\
 &\quad + 3\mu T F_1 (1 - e_j F_2 \cos(\sigma_n + v_n)) S_{n+1}^2
 \end{aligned}$$

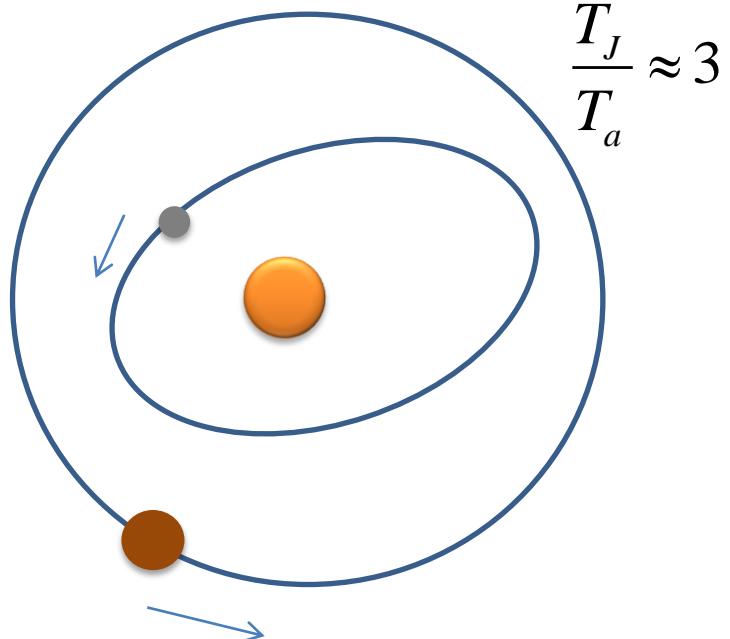
$$\begin{aligned}
 N_n &= N_{n+1} \\
 &\quad - \mu e_j T \sqrt{2S_n + 1} (G \sin(\sigma_n + v_n) - D \sin(\sigma_n - v_n)) \\
 &\quad + \mu T e_j F_1 F_2 \sin(\sigma_n + v_n) S_{n+1}^3 - 4T\mu e_j^2 K \sin(2v_n)
 \end{aligned}$$

$$\begin{aligned}
 v_{n+1} &= v_n + \frac{4T\mu_1^2}{(N_{n+1} - S_{n+1})^3} - \frac{3}{2} T + \mu T \frac{b}{N_{n+1}^2} \cos(2\sigma_n)
 \end{aligned}$$

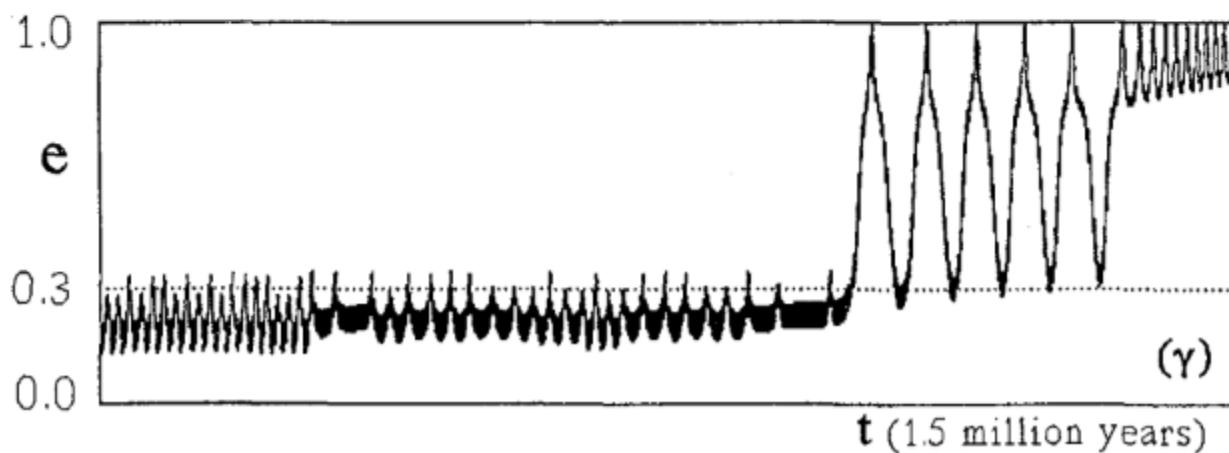
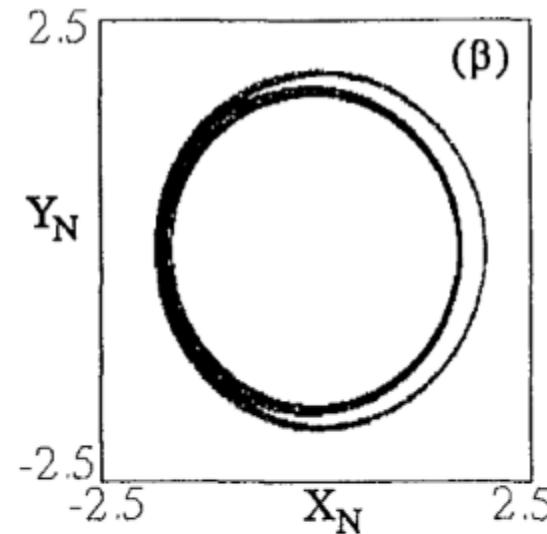
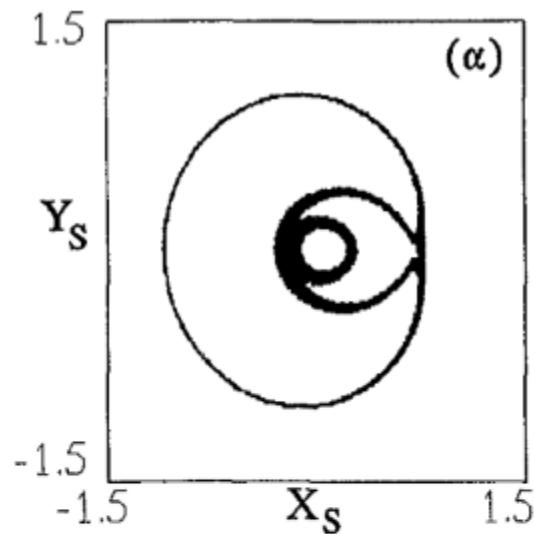
$$\begin{aligned}
 S &= \sqrt{\mu_1 a} (1 - \sqrt{1 - e^2}) & \sigma &= 0.5(3\lambda' - \lambda) - \omega \\
 N &= \sqrt{\mu_1 a} (3 - \sqrt{1 - e^2}) & v &= -0.5(3\lambda' - \lambda) + \omega'
 \end{aligned}$$

↓

$e = e(t)$, $a = a(t)$ $\Delta t \approx 12$ years



Hadjidemetriou's 3:1 resonance map



Escape
or
collision

<http://users.auth.gr/voyatzis/GVFTP/Dmaps/>

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