Approaching traveling water waves of large amplitude

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Joint work with A. Constantin and O. Scherzer.

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Historical background

- Stokes in 1847, irrotational
- Nekrasov, Levi-Civita, Struik in 1920s, small amplitude
- Krasovskiĭ in 1961, Keady, Norbury, Fraenkel, Toland, McLeod in 1978, large amplitude and stagnation point

- Gestner in 1802, Dubreil-Jacotin in 1934 rotational
- Peregrine, Da silva in 1988, Constantin, Escher, Strauss 2000's
- Ablowitz, Fokas 2006

Outline

- Derivation of the free Boundary Value Problem
- Transformation to a fixed nonlinear Boundary Value Problem
- Solutions of the Problem
- Figures

Preliminaries

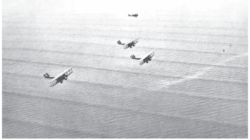
We restrict the problem in 2+1 dimensions (X, y, t).

- the velocity field of the flow $\big(U(X, y, t), V(X, y, t)\big)$
- the bottom $B = \{y = -d\}$
- the free boundary $S = \{y = \xi(X, t)\}$

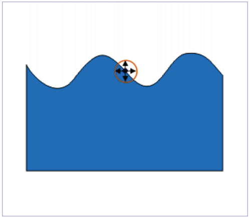
We study travelling water waves with propagation speed $c > 0$.
We introduce $x = X - ct$ such that

- $U(X, y, t) = u(x, y)$
- $V(X, y, t) = v(x, y)$
- $\xi(X, t) = \eta(x)$

Preliminaries - Illustration

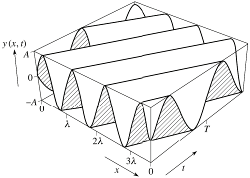


(a) Free boundary of travelling wave in 2+1 dimensions.



(b) Reducing one dimension.

Figure: Examples of travelling waves.



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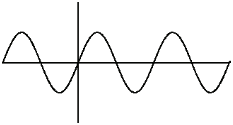


(b) Reducing one dimension.

The Euler Equations in 2-d

See *Constantin & Strauss, Comm. Pure & Appl. Math., 2004*.
Under the assumption that the water is inviscid, the two dimensional incompressible Euler equations, with density $\rho = 1$ become

$$\begin{aligned} u_x + v_y &= 0, \\ (u-c)u_x + vu_y &= -P_x, \\ (u-c)v_x + vv_y &= -P_y - g, \end{aligned}$$

where P is the pressure and g is the gravity.
We have the following boundary conditions

$$\begin{aligned} P &= P_{atm} \text{ on } S, \\ v &= (u-c)\eta_x \text{ on } S, \\ v &= 0 \text{ on } B. \end{aligned}$$

The flow is periodic: P, η and (u, v) have period 2π.

Stream function - Definition

We first define the relative mass flux by

$$p_0(x) := \int_{-d}^{\eta(x)} (u - c)dy < 0.$$

In fact, the relative mass flux is independent of x, that is, $p_0(x) = p_0$.
By virtue of the incompressibility condition, define the stream function $\psi(x, y)$ as the unique solution of the differential equation

$$\psi_x = -v, \qquad \psi_y = u - c \text{ in } \overline{\mathcal{D}},$$

which satisfies

$$\psi(x, -d) = -p_0 .$$

Thus

$$\Delta\psi = -\omega,$$

where $\omega = v_x - u_y$.

Free Boundary Value Problem 1/2

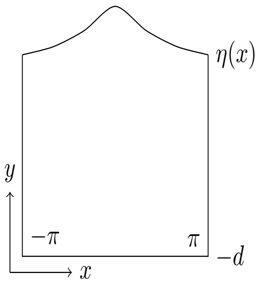
See *Constantin, K & Scherzer, SIAM Appl. Math., 2015*.

The constants

g (gravity), p_0 (relative mass flux),
Q (hydraulic head) and the function
$\gamma : [p_0, 0] \mapsto \mathbb{R}$ (vorticity) are given.

Moreover,

for given η, which we assume to be normalized to satisfy $\int_{-\pi}^{\pi} \eta(x)dx = 0$,
let $\psi = \psi[\eta]$ be the solution of



$$\Delta\psi = \gamma(\psi),$$

with boundary conditions

$$\psi(x, -d) = -p_0, \text{ on } B \quad \text{and} \quad \psi = 0 \text{ on } S,$$

$$\psi(\pi, y) = \psi(-\pi, y) \quad \text{and} \quad \psi_x(\pm\pi, y) = 0, \qquad \text{for } y \in [-d, \eta(x)].$$

Free Boundary Value Problem 2/2

For given η this *linear* PDE is overdetermined by imposing the non-linear boundary condition, known as the Bernoulli's law

$$\mathcal{B}_B[\psi] := |\nabla\psi|^2 + 2g(\eta(x) + d) = Q, \text{ on } S.$$

The free boundary problem consists in using the over-determinacy to determine η.

The free boundary value problem can also be viewed as solving an operator equation

$$\mathcal{G}(\eta) = 0\,,$$

where $\mathcal{G} : \eta \mapsto \mathcal{B}_B[\psi[\eta]]$.

The vorticity function

Using the condition $u < c$, we will show that there exists a function γ, s.t.

$$\omega = -\gamma(\psi).$$

$$P_{xy} = P_{yx}$$

in the Euler equations yield

$$-c(u_{xy} - v_{xx}) + u(u_{xy} - v_{xx}) + v(u_{yy} - v_{xy}) = 0$$

or

$$-c(u_y - v_x)_x + u(u_y - v_x)_x + v(u_y - v_x)_y = 0.$$

Use $\omega = v_x - u_y$.

The vorticity function

Use $\omega = v_x - u_y$

$$c\omega_x - u\omega_x - v\omega_y = 0.$$
$$(c-u)\omega_x - v\omega_y = 0.$$

The vorticity function

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The vorticity function

Use $\omega = v_x - u_y$

$$c\omega_x - u\omega_x - v\omega_y = 0.$$
$$(c - u)\omega_x - v\omega_y = 0.$$

Recall from the definition of the stream function

$$\psi_x = -v, \qquad \psi_y = u - c.$$

$$-\psi_y\omega_x + \psi_x\omega_y = 0$$

By the condition $u < c$, we get

$$\nabla\psi \parallel \nabla\omega.$$

Hence, ψ and ω have the same level set and $\omega = -\gamma(\psi)$.

Boundary Conditions

We have

$$\psi|_S - \psi|_B = \int_{-d}^{\eta(x)} \psi_y dy = \int_{-d}^{\eta(x)} (u - c) dy = p_0.$$

Thus

$$\psi = 0 \text{ on } S \, .$$

Finally, because the function ψ is periodic on $(-\pi, \pi)$ and even it follows that

$$\psi(\pi, y) = \psi(-\pi, y) \text{ for } y \in [-d, \eta(x)]$$

and

$$\psi_x(\pm\pi, y) = 0 \text{ for } y \in [-d, \eta(x)].$$

From the literature the evenness of ψ reflects the requirement that u and η are symmetric while v is antisymmetric around the line located strictly below the wave crest $x = 0$. Any solution with a free surface S that is monotone between crest and trough has to be symmetric

Boundary Conditions - Bernoulli's law

The Euler's equations read

$$-P_y = (u-c)u_y + vv_y + g + (u-c)\omega$$
$$-P_x = (u-c)u_x + vv_x - v\omega$$

Equivalently

$$\frac{(u-c)^2+v^2}{2} + gy + P(x,y) + \Gamma(-\psi) = \text{constant},$$

where

$$\Gamma(\psi) = \int_0^{\psi} \gamma(-s)ds\,.$$

Introducing the constant

$$Q = 2(\text{constant} + gd - P_{\text{atm}})\,,$$

called the *hydraulic pressure* we get that

$$\begin{aligned}\mathcal{B}_B[\psi] :=& |\nabla\psi|^2 + 2g(y+d) - Q \\ =& v^2 + (u-c)^2 + 2g(\eta(x)+d) - Q = 0 \text{ on } S\,.\end{aligned}$$

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Dubreil-Jacotin transformation

See *Constantin & Strauss, Comm. Pure & Appl. Math., 2004*.
Since, $\psi(x, y)$ is

- constant in B and S
- strictly decreasing in y

the height h above the flat bottom is a single-valued function of ψ. Let

$$q = x, \qquad p = -\psi(x, y).$$

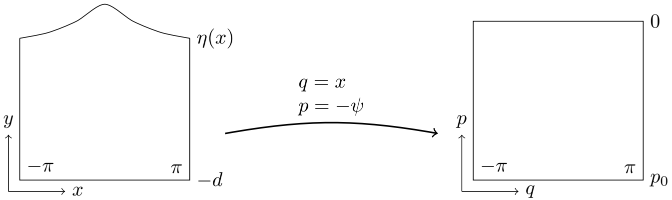

Figure: Dubreil-Jacotin transformation

The height function

Define

$$h(q,p) = y + d.$$

Then

$$h_q = \frac{v}{u-c}, \quad h_p = \frac{1}{c-u}, \qquad\qquad v = -\frac{h_q}{h_p}, \quad u = c - \frac{1}{h_p}$$

and

$$\partial_q = \partial_x - \frac{v}{u-c}\partial_y, \quad \partial_p = \frac{1}{c-u}\partial_y, \qquad\qquad \partial_x = \partial_q - \frac{h_q}{h_p}\partial_p, \quad \partial_y = \frac{1}{h_p}\partial_p.$$

The height function

Define

$$h(q,p) = y + d.$$

Then

$$h_q = \frac{v}{u-c}, \quad h_p = \frac{1}{c-u}, \qquad v = -\frac{h_q}{h_p}, \quad u = c - \frac{1}{h_p}$$

and

$$\partial_q = \partial_x - \frac{v}{u-c}\partial_y, \quad \partial_p = \frac{1}{c-u}\partial_y, \qquad \partial_x = \partial_q - \frac{h_q}{h_p}\partial_p, \quad \partial_y = \frac{1}{h_p}\partial_p.$$

Using the main equation derived from the Euler equations with the above formula for ∂_q we find

$$\omega_q = \omega_x - \frac{v}{u-c}\omega_y = 0, \qquad u < c$$

i.e.

$$\omega = -\gamma(-p).$$

The height equation

Now compute $-\gamma(-p) = \omega = v_x - u_y$ with usage of the above equations

$$
\begin{aligned}
-\gamma(-p) = \partial_x v - \partial_y u &= \left(\partial_q - \frac{h_q}{h_p}\partial_p\right)\left(-\frac{h_q}{h_p}\right) - \frac{1}{h_p}\partial_p\left(c - \frac{1}{h_p}\right) \\
&= \frac{-h_p h_{qq} + h_q h_{pq}}{h_p^2} - \frac{-h_p h_q h_{pq} + h_q^2 h_{pp}}{h_p^3} - \frac{h_{pp}}{h_p^3}.
\end{aligned}
$$

Moreover,

$$
|\nabla\psi|^2 = v^2 + (u-c)^2 = \frac{1 + h_q^2}{h_p^2}.
$$

The height equation

The boundary value problem

$$\begin{aligned}
\mathcal{H}[h] &:= (1+h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma(-p)h_p^3 = 0, && (q,p) \in D \\
\mathcal{B}_0[h] &:= 1 + h_q^2 + (2gh - Q)h_p^2 = 0, && p = 0, \\
\mathcal{B}_1[h] &:= h = 0, && p = p_0,
\end{aligned}$$

where D is the rectangle $(-\pi,\pi) \times (p_0, 0)$,
h even and 2π-periodic in q.

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Laminar solutions

Begin with the laminar flow, i.e. independent of q. Then the BVP becomes

$$\begin{aligned} &H_{pp} - \gamma(-p)H_p^3 = 0, && p_0 < p < 0, \\ &1 + (2gH - Q)H_p^2 = 0, && p = 0, \\ &H = 0, && p = p_0. \end{aligned}$$

Hence,

$$H(p) = \int_{p_0}^{p} \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}},$$

where $\Gamma(s) = \int_0^s \gamma(-p)dp$ and λ satisfies the integral equation

$$Q = \lambda + 2g \int_{p_0}^{0} \frac{dp}{\sqrt{\lambda - 2\Gamma(p)}}.$$

Linearized solutions - Variational formulation

For $\gamma(-p) = \gamma$, constant, linearise the system of equations around $H(p)$:

$$0 = \frac{\delta \mathcal{H}}{\delta H} := \frac{d}{d\epsilon} \mathcal{H}[H + \epsilon m]\Big|_{\epsilon=0}.$$

We get

$$\begin{aligned} & m_{pp} + H_p^2 m_{qq} = 3\gamma H_p^2 m_p, && (q,p) \in D, \\ & gm = \lambda^{3/2} m_p, && p = 0, \\ & m = 0, && p = p_0, \end{aligned}$$

where m is even and 2π-periodic in q.
Under some explicit condition, there exist

- $\lambda^* > 2\Gamma_{max} = 2 \max_{p \in [p_0, 0]} \Gamma(p) > 0$
- solution $m(q, p)$.

Bifurcation

This specific value of λ^* defines $Q^* = Q(\lambda^*)$ and an explicit integral representation of $H^*(p) = H(p; \lambda^*)$.
Now, let

$$\mathcal{T} = \left\{ \left(Q(\lambda), H(p; \lambda) \right), \lambda > -2\Gamma_{min} \right\}$$

be the curve of the laminar flows.

It is shown that (Q^*, H^*) is a bifurcation point.

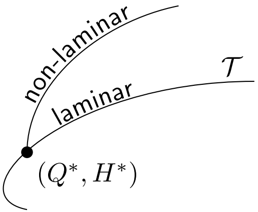


Figure: Bifurcation on the curve of laminar flows.

Solutions - The irrotational case

For the irrotational case the perturbed solution is given by

$$h(q,p;b) = H^*(p) + b\cos q\ M(p),$$

where the laminar flow H^* is obtained analytically, i.e.,

$$H^*(p) = \frac{p-p_0}{\sqrt{\lambda^*}}$$

and Q^* is readily obtained by

$$Q^* = \lambda^* - \frac{2gp_0}{\sqrt{\lambda^*}},$$

where $\lambda^* > 0$ is the solution of

$$\lambda + g\tanh\left(\frac{p_0}{\sqrt{\lambda}}\right) = 0.$$

Moreover,

$$M(p) = \sinh\left(\frac{p-p_0}{\sqrt{\lambda^*}}\right)$$

Solutions - The constant vorticity case

When $\gamma =$constant, the laminar flow H^* is given by

$$H^*(p) = \frac{\sqrt{\lambda^* - 2p\gamma} - \sqrt{\lambda^* - 2p_0\gamma}}{\gamma}$$

and

$$Q^* = \lambda^* - \frac{4gp_0}{\sqrt{\lambda^*} + \sqrt{\lambda^* - 2p_0\gamma}},$$

where $\lambda^* > 0$ is the solution of the equation

$$\tanh\left(\frac{\sqrt{\lambda} - \sqrt{\lambda - 2p_0\gamma}}{\gamma}\right) = \frac{\lambda}{g - \gamma\sqrt{\lambda}}.$$

Moreover,

$$M(p) = \frac{1}{\sqrt{\lambda^* - 2p\gamma}} \sinh\left(\frac{\sqrt{\lambda^* - 2p\gamma} - \sqrt{\lambda^* - 2p_0\gamma}}{\gamma}\right).$$

Linearized solutions - Asymptotic form

For $\gamma(-p)$ =constant, we consider a parametrized family of functions of the form

$$\hat{h}(q,p) = h_0(p) + bh_1(q,p), \qquad \text{for } b \in \mathbb{R}\,,$$

where h_0 is the laminar flow, i.e the q-independent solution. Formulate and find the solution of the BVP for h_1 such that

$$\mathcal{H}[\hat{h}](p,q) = \mathcal{O}(b^2)\,,\ \mathcal{B}_0[\hat{h}](q) = \mathcal{O}(b^2) \text{ and } \mathcal{B}_1[\hat{h}](q) = 0\,.$$

Higher order solutions

For $\gamma(-p)$ =constant, we consider a parametrized family of functions of the form

$$\hat{h}(q,p) = h_0(p) + bh_1(q,p) + b^2 h_2(q,p), \qquad \text{for } b \in \mathbb{R}\,,$$

where h_0 is the laminar flow, i.e the q-independent solution. We analytically determine the explicit formulas for h_1 and h_2 such that

$$\mathcal{H}[\hat{h}](p,q) = \mathcal{O}(b^3)\,,\ \mathcal{B}_0[\hat{h}](q) = \mathcal{O}(b^3) \text{ and } \mathcal{B}_1[\hat{h}](q) = 0\,.$$

Higher order- Equation and solutions

See *Constantin, K & Scherzer, NonL.Anal.-Real World Appl., 2015*.
h_2 satisfies the following linear BVP

$$\begin{aligned}
&(h_2)_{pp} + H_p^2 (h_2)_{qq} - 3\gamma H_p^2 (h_2)_p = \mathcal{P}_1[h_0, h_1] = \text{known}, && (q,p) \in D,\\
&(h_2)_p - g H_p^3 h_2 = \mathcal{P}_2[h_0, h_1] = \text{known}, && p = 0,\\
&h_2 = 0, && p = p_0,
\end{aligned}$$

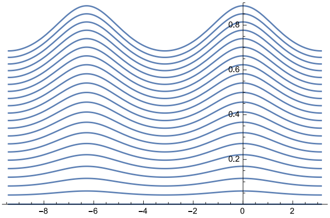
where h_2 is even and 2π-periodic in q.

For $\gamma = 0$ we get the formula

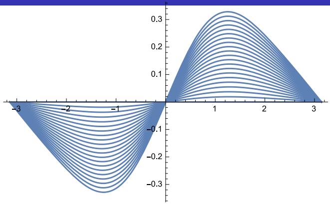
$$h_2(q,p) = A \sinh\left(2\frac{p-p_0}{\sqrt{\lambda}}\right)\cos(2q) + \frac{1}{4}\sinh\left(2\frac{p-p_0}{\sqrt{\lambda}}\right) + B(p-p_0).$$

For $\gamma \neq 0$ the formula is of the same form.

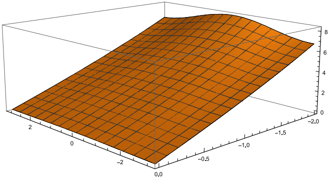
Figures



(a) The height along the streamlines for two periods.



(b) The vertical velocity v along the streamlines for one period.



(c) The water pressure for one period.

Figure: Constant vorticity $\gamma = -1.5$.

Generalization - Definitions

Define the approximation for the hydraulic head of the flow,

$$Q \approx Q^{(2N)}(b) = Q^* + \sum_{k=1}^{N} Q_{2k} b^{2k}, \qquad b \in \mathbb{R}. \tag{1}$$

Define the approximation for the height function $h(q, p; Q)$,

$$h(q, p; Q) \approx h^{(2N+1)}(q, p; b) = \sum_{n=0}^{2N+1} h_n(q, p) b^n, \tag{2}$$

with

$$h_{2k}(q, p) = \sum_{m=0}^{k} \cos(2mq) f_{2m}^{2k}(p)$$

and

$$h_{2k+1}(q, p) = \sum_{m=0}^{k} \cos((2m+1)q) f_{2m+1}^{2k+1}(p),$$

where $f_0^0(p) = H(p; \lambda_*)$.

Generalization - Theorem (*K, submitted*)

Let g, p_0, γ be fixed. Let λ_* be defined as the solution of the equation

$$\tanh\left(\frac{\sqrt{\lambda}-\sqrt{\lambda-2p_0\gamma}}{\gamma}\right)=\frac{\lambda}{g-\gamma\sqrt{\lambda}}$$

and Q^* given by

$$Q^*=\lambda^*-\frac{4gp_0}{\sqrt{\lambda^*}+\sqrt{\lambda^*-2p_0\gamma}}.$$

They exist specific sets of functions $\{h_n(q,p)\}_{n=1}^{2N+1}$ and constants $\{Q_{2k}\}_{k=1}^{N}$, such that the function $h^{(2N+1)}(q,p;b)$ defined in (2) is satisfying the system

$$\mathcal{H}[h^{(2N+1)}](q,p)=\mathcal{O}(b^{2N+2})\,,$$
$$\mathcal{B}_0[h^{(2N+1)}](q)=\mathcal{O}(b^{2N+2})\quad \text{and}\quad \mathcal{B}_1[h^{(2N+1)}](q)=0\,,$$

under the constraint that the hydraulic head Q is given by (1).

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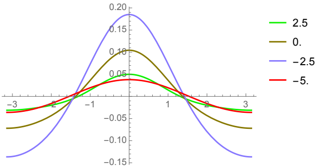
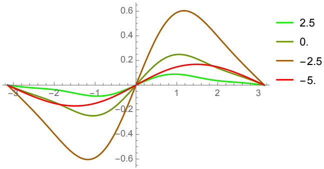
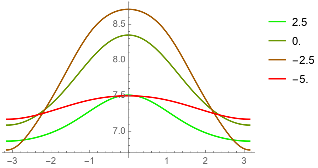
Figures - On the free boundary



(a) The free boundary $\eta(x)$.



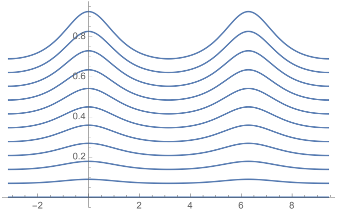
(b) The vertical velocity v .



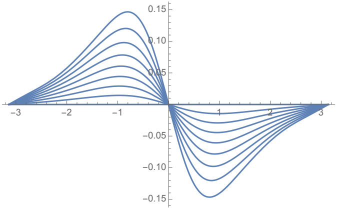
(c) The water pressure on the flat the bottom.

Figure: For different values of constant vorticity γ.

Figures - Fifth order approximation



(a) The height of the water h.



(b) The vertical velocity v .

Figure: On the streamlines.

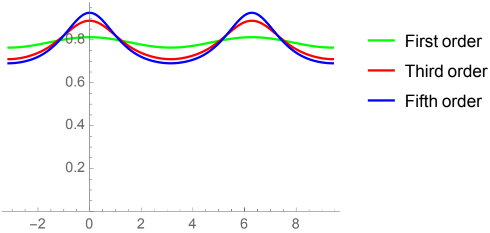
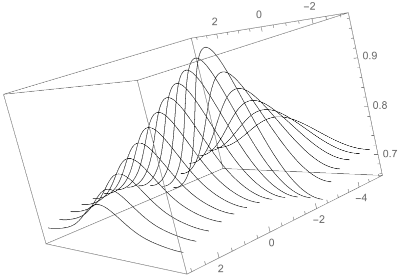


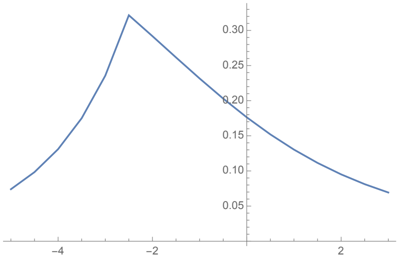
Figure: The free boundary.

Work in progress and future plans

- Non-constant vorticity.
- Construction of an algorithm based on the expansion for h and Q.
- Iterative algorithms with initial guess the approximate solution $\hat{h}$.



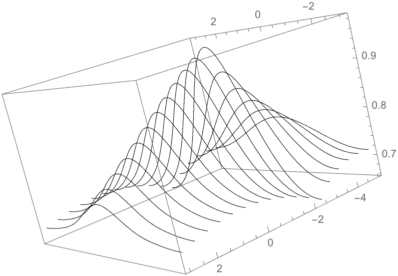
(a) Free boundaries.



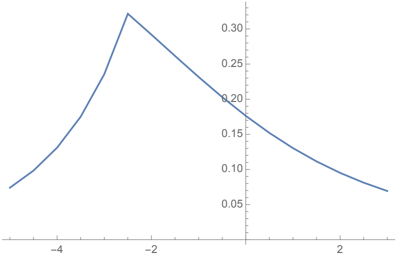
(b) The wave height.

Figure: For different values of vorticity

Figures



(a) Free boundaries.



(b) The wave height.

Figure: For different values of vorticity

Thank you for your attention.