Experiments of Boundary Stimulations of Social Influence Networks

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Scripts and simulations at:
https://github.com/mboudour/SocInlfluenceSims

- **The IBV Problem of Diffusion**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha \Delta u, x \in \Omega, t > 0, \\ u(x,t) &= f(x,t), x \in \partial\Omega, t > 0, \\ u(x,0) &= g(x), x \in \Omega \text{ (where } f(x,0) = g(x), x \in \partial\Omega). \end{aligned}$$

- **Discretization in Time and Space**

$$\begin{aligned} u(x, t+\delta t) &= u(x,t) + u_t(x,t)\delta t + O(\delta t^2), \\ u(x+\delta x, t) &= u(x,t) + u_x(x,t)\delta x + \frac{1}{2}u_{xx}(x,t)\delta x^2 + O(\delta x^3), \\ u(x-\delta x, t) &= u(x,t) - u_x(x,t)\delta x + \frac{1}{2}u_{xx}(x,t)\delta x^2 + O(\delta x^3), \\ u_t(x,t) &= \frac{u(x,t+\delta t) - u(x,t)}{\delta t} + O(\delta t), \\ u_{xx}(x,t) &= \frac{u(x-\delta x,t) - 2u(x,t) + u(x+\delta x,t)}{\delta x^2} + O(\delta x^2). \end{aligned}$$

- **The Case of One-Dimensional Lattice**

$$
\begin{aligned}
\Omega &= \{x_j : j = 1, \ldots, n\}, \\
\partial\Omega &= \{x_1, x_n\}, \\
t &\in \{t_m : m = 0, 1, \ldots\}.
\end{aligned}
$$

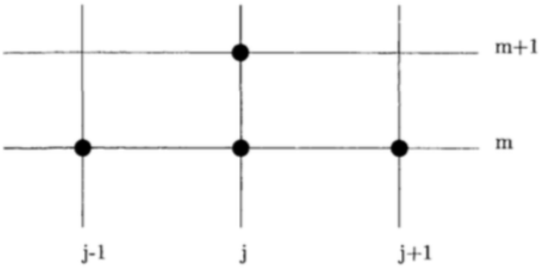


FIGURE 4.1. *The computational molecule of the explicit scheme.*

- **Notation**

$$
\begin{aligned}
x_j &= j, \quad j = 1, \ldots, n, \\
t_m &= m, \quad m = 0, 1, \ldots, \\
u(x_j, t_m) &= u_j^m, \\
f(x_j, t_m) &= f_j^m, \\
g(x_j) &= g_j.
\end{aligned}
$$

- **The Parameter** r

$$r = \alpha \frac{\delta t}{\delta x^2}.$$

The One-Dimensional Lattice Ω as a Graph

- **The Discrete Equation of Diffusion**

$$\begin{aligned} u_j^{k+1} - u_j^k &= r(u_{j-1}^k + u_{j+1}^k) - 2ru_j^k, j \in \Omega, k = 0, 1, \ldots, \\ u_j^k &= f_j, j \in \partial\Omega, k = 0, 1, \ldots, \\ u_j^0 &= g_j, j \in \Omega. \end{aligned}$$

- **Stability Condition**

$$r < \frac{1}{2}$$

- Adjacency between nodes of Ω

$$i \sim j \Leftrightarrow |i - j| = 1.$$

- **Adjacency Matrix**

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

- **Degrees of nodes of Ω**

$$\deg_i = \sum_{j=1}^{n} A_{ij} = \begin{cases} 2, & \text{for } i = 2, \ldots, n-1, \\ 1, & \text{for } i = 1, n. \end{cases}$$

- **Notation**
  - Let D denote the diagonal **degree matrix** such that $D_{ii} = \deg_i$.
  - Let I denote the diagonal **unit matrix** such that $I_{ii} = 1$.

- Thus, the discrete equation of diffusion in vector form is written as:

$$\begin{aligned} u^{k+1} - u^k &= r\big(A - D\big)u^k = \\ &= -rLu^k, \text{ in } \Omega, k = 0, 1, \ldots, \\ u^k &= f, \text{ on } \partial\Omega, k = 0, 1, \ldots, \\ u^0 &= g, \text{ in } \Omega. \end{aligned}$$

where the $n \times n$ matrix $L = D - A$ denotes the (**combinatorial**) **Laplacian matrix** $L = L_{ij}$ of graph Ω:

$$L_{ij} = \begin{cases} \deg_i, & \text{whenever } i = j, \\ -1, & \text{whenever } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

The Discrete Equation of Diffusion on a Graph G

- Let $G = (V, E)$ be a (general) graph, such that the set of vertices V is partitioned in two subsets:

$$V = \Omega \cup \partial\Omega,$$

assuming that

$$\partial\Omega = \{x \notin \Omega \colon \exists y \in S \text{ such that } y \sim x\}.$$

- The **discrete equation of diffusion on G with Dirichlet boundary conditions** is (in vector form):

$$\begin{aligned} u^{k+1} - u^k &= -rLu^k, \text{ in } \Omega, k = 0, 1, \dots, \\ u^k &= 0, \text{ on } \partial\Omega, k = 0, 1, \dots, \\ u^0 &= g, \text{ in } \Omega. \end{aligned}$$

where L is the Laplacian matrix of G.

The DeGroot–Friedkin–Johnsen Model of Social Influence

- Let $G = (V, E)$ be a **graph** of n persons, i.e., we assume form now on that **vertices** $\equiv$ **persons**.
- For each person $i \in V$ and time $k = 0, 1, 2, \ldots$ (in the discrete case considered here), let $v_i^k \in \mathbb{R}$ denote i's **opinion** at time k.
- Person's i opinion at time t is updated at next instance $t + 1$ according to the following equation of the **DeGroot–Friedkin–Johnsen Model of Social Influence**:

$$v_i^{k+1} = s_i \, N \, v_i^k + (1 - s_i) \, v_i^0,$$

  - where $N \, v_i^t$ is the average opinion of i's neighbors
  - and s_i is person's i **susceptibility coefficient**, a scalar parameter in the interval $(0, 1]$.

1. Remarks on the defition of the **susceptibility coefficient**:
   - If $s_i = 0$, then i's opinion does not change ($v_i^k = v_i^0$, for each time $k = 1, 2, \ldots.$). Such a person is called **stubborn** or **persistent** in her opinion.
   - If $s_i = 1$, then i adopts the average opinion of her neighbors $N\, v_i^t$. Such a person is called **malleable** or **fully compliant** in adopting her neighbor's influence.
   - If $0 < s_i < 1$, then i's opinion is inserted in-between $N\, v_i^t$ and v_i^0, where the exact inserted position is weighted by s_i.
2. Remaks on the definition of matrix N, which is called **walk matrix** on graph G:
   - Denoting by A, D the adjacency and the degree matrix of G, we have
     $$N = D^{-1}A.$$
   - Moreover, denoting by L, I the Laplacian and the unit matrix of G, we have
     $$N = I - D^{-1}L.$$

Reduction of Social Influence to a Diffusion Process

- Denoting by S the $n \times n$ diagonal matrix with its diagonal entries equal to the s_i's, if

$$S = I$$

(i.e., if all persons are fully malleable), then the DeGroot–Friedkin–Johnsen model of social influence becomes (in vector form):

$$v^{k+1} = N\, v^k,$$

i.e., since $N = I - D^{-1}L$,

$$v^{k+1} - v^k = -D^{-1}\, L\, v^k,$$

which is a diffusion equation with a **variable diffusion coefficient** equal to D^{-1}.

Reduction of Diffusion Process to Social Influence

- If G is a d–regular graph, for a sufficiently large positive integer d, and

$$r = \frac{1}{d},$$

then the equation of a diffusion process (in vector form) becomes:

$$u^{k+1} - u^k = -D^{-1}Lu^k = Nu^k - u^k,$$

i.e.,

$$u^{k+1} = Nu^k,$$

which is a process of social influence (in the DeGroot–Friedkin–Johnsen model) for $S = I$ (i.e., when all persons are fully malleable).

The Boundary of a Social Influence Process

- Suppose that we have a process of social influence on a graph $G = (V, E)$ of n persons indexed by $i = 1, 2, \ldots, n$ such that each person has a coefficient of susceptibility $s_i \in [0, 1]$.
- Assumptions and Notation:
  - G is connected.
  - $V = \Omega \cup \partial\Omega$, where $\Omega \cap \partial\Omega = \varnothing$.
  - $\Omega = \{i \in V : s_i > 0\}$.
  - $\partial\Omega = \{i \in V : s_i = 0\}$.
  - $|\partial\Omega| < n$.
  - $\forall j \in \delta\Omega, \exists i \in \Omega, i \sim j$.

Sources of Boundary Stimulations to Social Influence

- Given a process of social influence on a graph $G = (\Omega \cup \partial\Omega, E)$ with regards to a vector of susceptibility coefficients $s = \{s_i\}_{i \in \Omega}$, person $j_b \in \partial\Omega$ is said to be a **source of boundary stimulation**, or a **persistent source**, if j_s fires up the process governed by the following IBVP:

$$\begin{aligned} v_i^{k+1} &= s_i \, N \, v_i^k + (1 - s_i) v_i^0, \text{ for } i \in \Omega, k = 0, 1, \ldots, \\ v_{j_b}^k &= 1, \text{ for } k = 0, 1, \ldots, \\ v_i^0 &= \phi_i, \text{ for } i \in \Omega, \end{aligned}$$

where ϕ_i is the initial opinion of person $i \in \Omega$.

- Note that, for the zero initial contidion $\phi_i = 0, i \in \Omega$, the IBVP of social influence becomes:

$$
\begin{aligned}
v_i^{k+1} &= s_i\, N\, v_i^k + (1 - s_i)\psi_i, \text{ for } i \in \Omega, k = 1, 2, \ldots, \\
v_{j_b}^k &= 1, \text{ for } k = 1, 2, \ldots, \\
v_i^1 &= \psi_i, \text{ for } i \in \Omega,
\end{aligned}
$$

where

$$
\psi_i = \begin{cases} \frac{s_i}{\deg_i}, & \text{whenever } i \sim j_b, \\ 0, & \text{whenever } i \nsim j_b. \end{cases}
$$

Transient and Steady Solutions to Boundary Stimulations

- Let $v_i^k(j_b) = v_i^k(j_b; s)$ be the **transient** solution to the IBVP of social influence stimulated by source j_b with regards to the vector of susceptibility coefficients $s = \{s_i\}_{i \in \Omega}$.
- Then, for any $i \in \Omega$, the following limit exists:

$$\lim_{k \to \infty} v_i^k(j_b; s) = \overline{v}_i(j_b; s).$$

- $\overline{v}(j_b) = \overline{v}(j_b; s) = \{\overline{v}_i(j_b; s)\}_{i \in \Omega}$ is the **steady state** solution to the BV of social influence on G, which is stimulated by source j_s with regards to s:

$$\overline{v}(j_b) = (I - SN)^{-1} (I - S) \psi,$$

where $\psi = \{\psi_i\}_{i \in \Omega}$.
- If, for all $i \in \Omega$, $s_i = 1$, then

$$\lim_{k \to \infty} v_i^k(j_b; \mathbf{1}) = 1, \text{ for any } i \in \Omega.$$

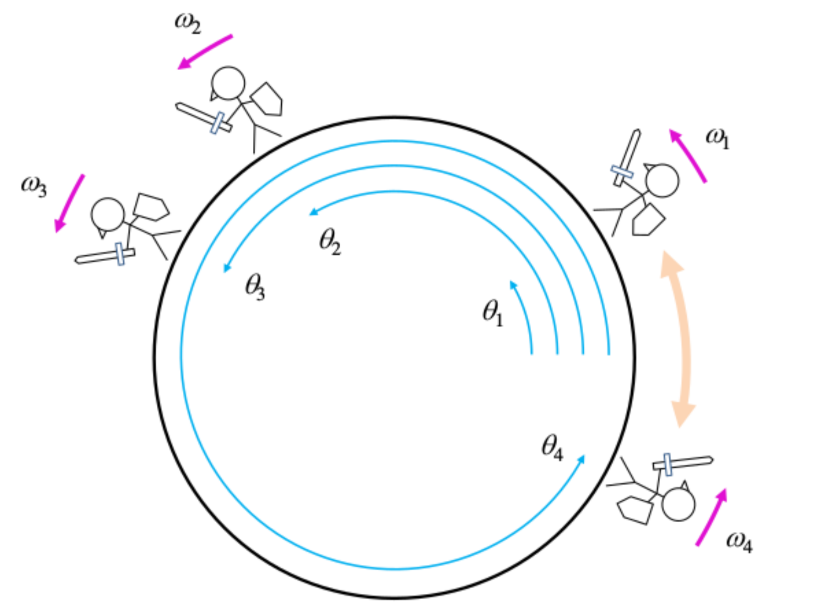
Synchronization from One Source of Alternating Boundary Simulation to Social Influence

- A person $j_a \in \partial\Omega$ is said to be an **alternating source of boundary stimulation**, or an **alternating source**, if j_a fires up a process of social influence on graph G that satisfies the following IBVP:

$$\begin{aligned} v_i^{k+1} &= s_i\, N\, v_i^k + (1 - s_i)\, v_i^0, \text{ in } \Omega, k = 0, 1, \ldots, \\ v_{j_a}^k &= 1, k = 0, 2, 4, \ldots, \\ v_{j_a}^k &= -1, k = 1, 3, 5, \ldots, \\ v_i^0 &= 0, \text{ in } \Omega. \end{aligned}$$

- Note that the opinion of the source j_a constantly oscillates between $+1$ and -1 without being influenced by her neighbors (although, as a source, j_a does influence her neighbors).

Synchronization in the Kuramoto Model



- **The Kuramoto Model of Synchronization among Coupled Oscillators on a Graph**

  Oscillators are located on the nodes of **graph** $G = (V, E)$. Each oscillator $i \in V$ marches on a circular track with a preferred speed ω_i. The position of oscillator i on the circle is given by an angle θ_i. However, since oscillators are considered to be **coupled** with regards to their graph proximities, i's angular position θ_i depends on her neighbors' angular positions θ_j according to the following equation:

$$\begin{aligned} \theta_i^{k+1} - \theta_i^k &= \omega_i + \alpha \frac{\sum_{j \sim i} \sin(\theta_j^k - \theta_i^k)}{\deg_i}, \text{ in } \Omega, k = 0, 1, \ldots, \\ \theta_i^0 &= \phi_i, \text{ in } \Omega. \end{aligned}$$

The Influentiability Matrix of a Graph Influence Process of Boundary Stimulations

- By a **graph influence process**, we mean a process of social influence on a graph $G = (V, E)$ with regards to a vector of susceptibility coefficients $s = \{s_i\}_{i \in V}$.
- The **influentiability matrix** of a graph influence process stimulated on the boundary of graph G is defined as the following $n \times n$ matrix $\mathcal{U}^\infty = \{\mathcal{U}_{ij}^\infty(s)\}_{i,j \in V}$:

$$\mathcal{U}_{ij}^\infty(s) = \begin{cases} 1, & \text{for } i = j, \\ \overline{v}_j(i; s), & \text{for } i \neq j. \end{cases}$$

The Influence Degree Centrality Index of a Node in a Graph Influence Process

- Given a graph influence process on a graph $G = (V, E)$ with regards to a vector of susceptibility coefficients $s = \{s_i\}_{i \in V}$, the **influence degree centrality index** of node $i \in V$ is defined as follows:

$$\begin{aligned} c_{influence-degree}(i; s) &= \frac{1}{n-1} \sum_{j \sim i} \mathcal{U}^{\infty}_{i,j}(s) \\ &= \frac{1}{n-1} \sum_{j \sim i} \overline{v}_j(i; s). \end{aligned}$$

- Denoting by $\mathbf{1}$ the $\mathbb{R}^{|V|}$ vector having all its components equal to 1, we get

$$c_{influence-degree}(i; \mathbf{1}) = \frac{1}{n-1} \deg_i = c_{degree}(i),$$

where $c_{degree}(i)$ is the **degree centrality index** of node $i \in V$.

For the Karate network: